Global Existence of Weak Solutions to a Three-dimensional Fractional Model in Magneto-Elastic Interactions

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Abstract—**This paper delves into the global existence of weak solutions for a three-dimensional magnetoelastic interaction model. This model combines a fractional harmonic map heat flow with an evolution equation for displacement. By using the Faedo-Galerkin method, we successfully establish the global existence of weak solutions for this coupled system.**

Index Terms—**Fractional derivative, Landau-Lifshitz equation, ferromagnets, elasticity, weak solution.**

I. Introduction

We consider the following problem [23]:

$$\mathbf{m}_t = \nu \mathbf{m} \times \mathbf{H}_{\text{eff}} - \mu \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}). \tag{1}$$

$$\rho\ \mathbf{n}_{tt} - \operatorname{div}\Big(\mathcal{S}(\mathbf{n}) + \frac{1}{2}\mathcal{L}(\mathbf{m})\Big) = 0. \tag{2}$$

The first equation, denoted as equation (1), is the famous Landau-Lifschitz equation, extensively studied in references [9] and [12]. This equation was originally introduced to characterize the dynamics of micro-magnetic processes. The evolution equation for the displacement field is given by (2). The magnetization vector, $\mathbf{m}$, is a map from D to S^2 (the unit sphere of $\mathbb{R}^3$) and $\mathbf{m}_t$ is its derivative with respect to time. The symbol $\times$ represents the vector cross product in $\mathbb{R}^3$. We denote by $m_i, i = 1, 2, 3$ the components of $\mathbf{m}$. $\mathbf{H}_{\text{eff}}$ symbolizes the effective field and in this research we assume

$$\mathbf{H}_{\text{eff}} = -\Lambda^{2\alpha}\mathbf{m} - \boldsymbol{\ell}(\mathbf{m}, \mathbf{n}) \tag{3}$$

$\Lambda = (-\Delta)^{\frac{1}{2}}$ designate the square root of the Laplacian which could be explained through Fourier transformation [21]. In this approach, we use the Einstein summation convention for repeated indices and we are more concerned in the case $\alpha \in (1, \frac{3}{2})$.

The components of the vector $\boldsymbol{\ell}(\mathbf{m}, \mathbf{n})$ and the tensors $\mathcal{S}(\mathbf{n})$, $\mathcal{L}(\mathbf{m})$ are represented by

$$\ell_i = \lambda_{ijkl}(x) m_j \epsilon_{kl}(\mathbf{n}), \qquad i = 1, 2, 3.$$

$$S_{kl} = \sigma_{ijkl}(x)\epsilon_{ij}(\mathbf{n}) \text{ and } \mathcal{L}_{kl} = \lambda_{ijkl}(x) m_i m_j.$$

Here $\epsilon_{ij}(\mathbf{n}) = \frac{1}{2}(\partial_i n_j + \partial_j n_i)$ represents the components of the lnearized strain tensorc ϵ, $\lambda_{ijkl}(x) = \lambda_1(x)\delta_{ijkl} + \lambda_2(x)\delta_{ij}\delta_{kl} + \lambda_3(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, $\sigma_{ijkl}(x) = \tau_1(x)(\delta_{ijkl} - \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) + \tau_2(x)\delta_{ij}\delta_{kl}$ whith $\delta_{ijkl} = 1$ si $i = j = k = l$ and $\delta_{ijkl} = 0$ otherwise. $\sigma(x) = (\sigma_{ijkl}(x))$, the elasticity tensor is expected to fulfill the following symmetry property

$$\sigma_{ijkl}(x) = \sigma_{klij}(x) = \sigma_{jikl}(x)$$

and moreover the inequality

$$(\sigma_{ijkl}(x)\epsilon_{ij}\epsilon_{kl}) \geq \beta \sum |\epsilon_{ji}|^2 \tag{4}$$

holds for some $\beta > 0$.

Our investigation is focused on the existence of solutions for the non-linear integro-differential problem described by equations (1)and (2).In this context, we refer to the work presented in paper [7], which establishes the existence theorem for the general three-dimensional magnetoelastic problem. We aim to investigate the existence of global weak solutions for a three-dimensional fractional problem in the case where the parameters ν,μ and ρ are considered as variables bounded coefficients.
We quote some references on the subjects of magnetoelasticity ([1],[5],[6], [10], [11]) and viscoelasticity ([3], [4], [8], [9], [13], [15], [16]) that inspired this paper.

The following notation will be used consistently throughout this work: For D an open bounded domain of $\mathbb{R}^3$, we denote by $\mathbf{L}^p(D) = (L^p(D))^3$ and $\mathbf{H}^1(D) = (H^1(D))^3$ the classical Hilbert spaces equipped with the usual norm denoted by $\|.\|_{\mathbf{L}^p(D)}$ and $||.||_{\mathbf{H}^1(D)}$ (in general, the product functional spaces $(X)^3$ are all simplified to $\mathbf{X}$). For all $s > 0$, $W^{s,p}$ denotes the usual Sobolev space consisting all f such that

$$\|f\|_{W^{s,p}} := \|\mathcal{F}^{-1}\big(1 + |\cdot|^2\big)^{\frac{s}{2}}\big(\mathcal{F}f\big)(\cdot)\|_{L^p} < \infty$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse. Let $\dot{W}^{s,p}$ denote the corresponding homogeneous Sobolev space. When $p = 2$, $W^{s,p}$ corresponds to the usual Sobolev space H^s and we have

$$\|f\|_{\dot{H}^s} := \|\Lambda^s f\|_{L^2}$$

We proceed as follows: In the following section, we present the model on which we will work and we give a preliminary result. In section 3 we we recall some lemmas. In section 4, we present the main result that we will subsequently prove in section 5.

II. The model and preliminary results

This paper delves into the global existence of weak solutions in the spatial domain $D = (0, 2\pi)^d$, with periodic boundary conditions for the magnetization vector. We consider $d = 3$ and assume that $\nu = 0$, $\mu_0 \leq \mu(x) \leq \mu_1$, $a_0 \leq$

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$a(x) \leq a_1$, $\rho_0 \leq \rho(x) \leq \rho_1$ and $\lambda_0 \leq \lambda_{ijkl}(x) \leq \lambda_1$. The generic point of D is denoted by $x = (x_1, x_2, x_3)$. The system under consideration is as follows:

$$\begin{cases} \mathbf{m}_t = -\mu(x)\mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}) \\ \rho(x)\ \mathbf{n}_{tt} - \operatorname{div}\Big(\mathcal{S}(\mathbf{n}) + \dfrac{1}{2}\mathcal{L}(\mathbf{m})\Big) + \mathbf{h} = 0, \end{cases} \tag{5}$$

where $\mathbf{h}$ is a given external force. We impose the following initial conditions:

$$\mathbf{n}(\cdot,0) = \mathbf{n}_0,\ \ \mathbf{n}_t(\cdot,0) = \mathbf{n}_1,\ \ \mathbf{m}(\cdot,0) = \mathbf{m}_0,\ \ |\mathbf{m}_0| = 1 \text{ in } D, \tag{6}$$

with as a boundary condition for the displacement vector

$$\mathbf{n} = 0 \quad \text{on} \quad \Sigma := \partial D \times (0,T). \tag{7}$$

The double vector product in the first equation(5)presents the main obstacle to straightforward analysis. To overcome this challenge, we introduce an equivalent equation

$$\mathbf{m} \times \mathbf{m}_t = \mu(x)\mathbf{m} \times \mathbf{H}_{\text{eff}}. \tag{8}$$

Following a well-established approach (see [4]), we replace the first equation in system (5) with a quasilinear parabolic equation of the Ginzburg-Landau type.

$$\mathbf{m}_t^\varepsilon + a(x)\mu(x)\Lambda^{2\alpha}\mathbf{m}^\varepsilon + \mu(x)\boldsymbol{\ell}(\mathbf{m}^\varepsilon, \mathbf{n}^\varepsilon) + \frac{|\mathbf{m}^\varepsilon|^2 - 1}{\varepsilon}\mathbf{m}^\varepsilon = 0. \tag{9}$$

Here ε is a positive parameter and $\mathbf{m}^\varepsilon : D \times \mathbb{R}^+ \to \mathbb{R}^3$. The ε-penalization in (9) replaces the magnitude constraint $|\mathbf{m}| = 1$.

III. Main result

Now we define the solution in the weak sense of the problem (5)-(6)-(7).

Definition III.1. *Let* $\mathbf{m}_0 \in \mathbf{H}^\alpha(D), |\mathbf{m}_0| = 1$ *a.e.,* $\mathbf{n} \in \mathbf{H}_0^1(D)$, $\mathbf{n}_1 \in \mathbf{L}^2(D)$ *and* $\mathbf{h} \in \mathbf{L}^2(Q)$. *We say that the pair* $(\mathbf{m}, \mathbf{n})$ *is a weak solution of the problem* (5)-(6)-(7) *if:*

- *for all* $T > 0$, $\mathbf{m} \in L^\infty(0,T;\mathbf{H}^\alpha(D)), \mathbf{m}_t \in L^2(0,T;\mathbf{L}^2(D))$, $|\mathbf{m}| = 1$ *a.e.,* $\mathbf{n} \in L^2(0,T;\mathbf{H}_0^1(D))$ *and* $\mathbf{n}_t \in L^2(0,T;\mathbf{L}^2(D))$*;*
- *for all* $\boldsymbol{\varphi} \in \mathbf{C}^\infty(\overline{Q})$ *and* $\boldsymbol{\psi} \in \mathbf{H}_0^1(Q)$, *we have:*

$$\int_Q (\mathbf{m}_t \times \mathbf{m}) \cdot \boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t + \int_Q a(x)\mu(x)\Lambda^\alpha\mathbf{m} \cdot \Lambda^\alpha(\mathbf{m} \times \boldsymbol{\varphi})\ \mathrm{d}x\mathrm{d}t$$

$$+ \int_Q (\mu(x)\boldsymbol{\ell}(\mathbf{m},\mathbf{n}) \times \mathbf{m}) \cdot \boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t = 0$$

$$-\int_Q \rho(x)\mathbf{n}_t \cdot \boldsymbol{\psi}_t\ \mathrm{d}x\mathrm{d}t + \int_Q \Big(\mathcal{S}(\mathbf{n}) + \frac{1}{2}\mathcal{L}(\mathbf{m})\Big) \cdot \epsilon(\boldsymbol{\psi})\ \mathrm{d}x\mathrm{d}t$$

$$+ \int_Q \mathbf{h} \cdot \boldsymbol{\psi}\ \mathrm{d}x\mathrm{d}t = 0;$$

- $\mathbf{m}(0,x) = \mathbf{m}_0(x)$ *and* $\mathbf{n}(0,x) = \mathbf{n}_0(x)$ *in the trace sense;*
- *for all* $T > 0$, *we have:*

$$\begin{aligned} &\frac{a_0}{2}\int_D |\Lambda^\alpha \mathbf{m}(T)|^2\ \mathrm{d}x \\ &+ \int_Q |\mathbf{m}_t|^2\ \mathrm{d}x\mathrm{d}t + \tfrac{\rho_0}{2}\int_D |\mathbf{n}_t(T)|^2\ \mathrm{d}x \\ &+ \tfrac{\beta}{4}\int_D |\nabla\mathbf{n}(T)|^2\ \mathrm{d}x - \tfrac{\beta}{4}\int_Q |\nabla \boldsymbol{n}_t^{\varepsilon,N}|^2\ \mathrm{d}x\mathrm{d}t \\ &\leq \frac{a_1}{2}\int_D |\Lambda^\alpha\mathbf{m}_0|^2\ \mathrm{d}x + \frac{\rho_1}{2}\int_D |\mathbf{n}_1|^2\ \mathrm{d}x \\ &+ \tfrac{3\tau}{4}\int_D |\nabla\mathbf{n}_0|^2\ \mathrm{d}x + C(D,\beta,\lambda,\mathbf{h}), \end{aligned} \tag{10}$$

where $C(D,\beta,\lambda,\mathbf{h})$ *is a positive constant which depends only on* D, β, λ *and* $\mathbf{h}$.

The principal outcome of this paper can be summarized as follows.

Theorem III.2. *Let* $\alpha \in (1, \frac{3}{2})$, $\mathbf{m}_0 \in \mathbf{H}^\alpha(D)$ *such that* $|\mathbf{m}_0| = 1$ *a.e.,* $\mathbf{n}_0 \in \mathbf{H}_0^1(D)$, $\mathbf{n}_1 \in \mathbf{L}^2(D)$ *and* $\mathbf{h} \in \mathbf{L}^2(Q)$. *Then a weak solution for the problem, as defined in* $III.1$, *is guaranteed to exist.*

A detailed proof of Theorem III.2 will be presented in Section 5.

IV. Some technical lemmas

This section introduces several key lemmas that will play a crucial role in subsequent analyses throughout the paper. To get started, we need a handy result from Lions ([16], p. 57)

Lemma IV.1. *Assume* X, Y *et* Z *are three Banach spaces and satisfy* $X \subset Y \subset Z$ *where the injections are continuous with compact embedding* $X \hookrightarrow Y$ *and X, Z are reflexive. Denote*

$$D := \{v | v \in L^{p_0}(0,T;X), v_t = \frac{dv}{dt} \in L^{p_1}(0,T;Z)\}$$

where T is finite and $1 < p_i < \infty$, $i = 0, 1$.
Then D, equipped with the norm

$$||v||_{L^{p_0}(0,T;X)} + ||v_t||_{L^{p_1}(0,T;Z)},$$

is a Banach space and the embedding $D \hookrightarrow L^{p_0}(0,T;Y)$ *is compact.*

We'll also need another handy lemma from Lions ([16], p. 12).

Lemma IV.2. *Let* Θ *be a bounded open set of* $\mathbb{R}_x^d \times \mathbb{R}_t$, h_k *and* h *in* $L^q(\Theta), 1 < q < \infty$ *such that* $||f_k||_{L^q(\Theta)} \leq C, f_k \to f$ *a.e. in* Θ, *then* $f_k \rightharpoonup f$ *weakly in* $L^q(\Theta)$.

Here is another Lemma "fractional calculus" whose proof can be found in [21].

Lemma IV.3. *Suppose that* $p > q > 1$ *and* $\frac{1}{p} + \frac{s}{d} = \frac{1}{q}$. *Assume that* $\Lambda^s h \in L^q$, *then* $f \in L^p$ *and there is a constant* $C > 0$ *such that*

$$\|h\|_{L^p} \leq C\|\Lambda^s h\|_{L^q}.$$

We conclude with this lemma (the proof can be found in [12]).

Lemma IV.4. *If u and v belong to* $H_{per}^{2\alpha}(D) := \{u \in L^2(D)/\Lambda^{2\alpha}u \in L^2(D)\}$, *then*

$$\int_D \Lambda^{2\alpha}u \cdot v\ dx = \int_D \Lambda^\alpha u \cdot \Lambda^\alpha v\ dx.$$

V. Proof of Theorem III.2

A. The penalty problem

We consider for $\varepsilon > 0$ fixed parameter the following problem

$$\begin{cases} \mathbf{m}_t^\varepsilon + a(x)\mu(x)\Lambda^{2\alpha}\mathbf{m}^\varepsilon + \mu(x)\boldsymbol{\ell}(\mathbf{m}^\varepsilon, \mathbf{n}^\varepsilon) \\ + \dfrac{|\mathbf{m}^\varepsilon|^2 - 1}{\varepsilon}\mathbf{m}^\varepsilon = 0 \\ \rho(x)\ \mathbf{n}_{tt}^\varepsilon - \operatorname{div}\Big(\mathcal{S}(\mathbf{n}^\varepsilon) + \dfrac{1}{2}\mathcal{L}(\mathbf{m}^\varepsilon)\Big) + \mathbf{h} = 0, \end{cases} \tag{11}$$

with the initial and boundary conditions:

$$\mathbf{n}^\varepsilon(\cdot, 0) = \mathbf{n}_0,\ \ \mathbf{n}_t^\varepsilon(\cdot, 0) = \mathbf{n}_1,$$

$$\mathbf{m}^\varepsilon(\cdot, 0) = \mathbf{m}_0,\ \ |\mathbf{m}_0| = 1 \quad a.e. \ \text{ in } \ D,$$

$$\mathbf{n}^\varepsilon = 0 \quad \text{on} \quad \Sigma.$$

We apply the Faedo-Galerkin method: for $\{f_i\}_{i\in\mathbb{N}}$ an orthonormal basis of $L^2(D)$ consisting of all the eigenfunctions for the operator $\Lambda^{2\alpha}$ (the existence of such a basis can be proved as in [22], Ch.II)

$$\Lambda^{2\alpha} f_i = \alpha_i f_i, i = 1, 2, ...$$

under periodic boundary conditions, and $\{g_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $L^2(D)$ consisting of all the eigenfunctions for the operator $-\Delta$

$$\begin{cases} -\Delta g_i = \beta_i g_i, i = 1, 2, ... \\ g_i = 0 \quad \text{on} \quad \partial D. \end{cases}$$

and we consider the following penalized system in $Q = D \times (0, T)$

$$\begin{cases} \mathbf{m}_t^{\varepsilon,N} + a(x)\mu(x)\Lambda^{2\alpha}\mathbf{m}^{\varepsilon,N} + \mu(x)\boldsymbol{\ell}(\mathbf{m}^{\varepsilon,N}, \mathbf{n}^{\varepsilon,N}) \\ + \dfrac{|\mathbf{m}^{\varepsilon,N}|^2 - 1}{\varepsilon}\mathbf{m}^{\varepsilon,N} = 0 \\ \rho(x)\ \mathbf{n}_{tt}^{\varepsilon,N} - \operatorname{div}\Big(\mathcal{S}(\mathbf{n}^{\varepsilon,N}) + \dfrac{1}{2}\mathcal{L}(\mathbf{m}^{\varepsilon,N})\Big) + \mathbf{h}^N = 0, \end{cases} \tag{12}$$

where the vector $\mathbf{h}^N$ satisfies

$$\int_D \mathbf{h}^N(x,t) g_i(x)\ \mathrm{d}x = \int_D \mathbf{h}(x,t) g_i(x)\ \mathrm{d}x,$$

as well as the corresponding initial and boundary conditions:

$$\mathbf{n}^{\varepsilon,N}(\cdot, 0) = \mathbf{n}^N(\cdot, 0),\ \ \mathbf{n}_t^{\varepsilon,N}(\cdot, 0) = \mathbf{n}_t^N(\cdot, 0),$$

$$\mathbf{m}^{\varepsilon,N}(\cdot, 0) = \mathbf{m}^N(\cdot, 0), \ \text{in } D,$$

$$\mathbf{n}^{\varepsilon,N} = 0 \quad \text{on} \quad \Sigma = \partial D \times (0, T).$$

and

$$\int_D \mathbf{n}^N(x,0) g_i(x)\ \mathrm{d}x = \int_D \mathbf{n}_0(x) g_i(x)\ \mathrm{d}x$$

$$\int_D \mathbf{n}_t^N(x,0) g_i(x)\ \mathrm{d}x = \int_D \mathbf{n}_1(x) g_i(x)\ \mathrm{d}x,$$

$$\int_D \mathbf{m}^N(x,0) f_i(x)\ \mathrm{d}x = \int_D \mathbf{m}_0(x) f_i(x)\ \mathrm{d}x.$$

We are seeking for approximate solutions $(\mathbf{m}^{\varepsilon,N}, \mathbf{n}^{\varepsilon,N})$ to (12) under the form

$$\mathbf{m}^{\varepsilon,N} = \sum_{i=1}^N \mathbf{a}_i(t) f_i(x) \quad , \quad \mathbf{n}^{\varepsilon,N} = \sum_{i=1}^N \mathbf{b}_i(t) g_i(x),$$

where $\mathbf{a}_i$ and $\mathbf{b}_i$ are $\mathbb{R}^3$-valued vectors.

Multiplying each scalar of the first equation (12) by f_i and the second part by g_i, and integrating over the domain D, leads to a system of ordinary differential equations involving the unknowns $(\alpha_i(t), \beta_i(t)), i = 1, 2, ..., N$. Standard ordinary differential equations theory allows us to demonstrate the existence of local solutions to the problem, which can be extended to the interval $[0, T]$ using a priori estimates. For this, we multiply the first equation of (12) by $\mathbf{m}_t^{\varepsilon,N}$ and the second by $\mathbf{n}_t^{\varepsilon,N}$ integrating in D, we obtain

$$\begin{cases} \displaystyle\int_D |\mathbf{m}_t^{\varepsilon,N}|^2 \mathrm{d}x + \int_D a(x)\mu(x)\Lambda^{2\alpha}\mathbf{m}^{\varepsilon,N} \cdot \mathbf{m}_t^{\varepsilon,N} \mathrm{d}x \\ \displaystyle + \int_D \mu(x)\boldsymbol{\ell}(\mathbf{m}^{\varepsilon,N}, \mathbf{n}^{\varepsilon,N}) \cdot \mathbf{m}_t^{\varepsilon,N} \mathrm{d}x \\ \displaystyle + \frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_D (|\mathbf{m}^{\varepsilon,N}|^2 - 1)^2 \mathrm{d}x = 0 \\ \displaystyle \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_D \rho(x)|\mathbf{n}_t^{\varepsilon,N}|^2 \mathrm{d}x \\ \displaystyle - \int_{\partial D}\Big(\mathcal{S}(\mathbf{n}^{\varepsilon,N}) + \tfrac{1}{2}\mathcal{L}(\mathbf{m}^{\varepsilon,N})\Big)\Upsilon \cdot \mathbf{n}_t^{\varepsilon,N} \mathrm{d}x \\ \displaystyle + \int_D \Big(\mathcal{S}(\mathbf{n}^{\varepsilon,N}) + \frac{1}{2}\mathcal{L}(\mathbf{m}^{\varepsilon,N})\Big) \cdot \nabla\mathbf{n}_t^{\varepsilon,N} \mathrm{d}x \\ \displaystyle + \int_D \mathbf{h}^N \cdot \mathbf{n}_t^{\varepsilon,N} \mathrm{d}x = 0 \end{cases}$$

where Υ is the outer unit normal at the boundary ∂D. On the other hand (note that $\lambda_{ijkl}(x) = \lambda_{jikl}(x)$)

$$\begin{aligned} &\int_D \boldsymbol{\ell}(\mathbf{m}^{\varepsilon,N}, \mathbf{n}^{\varepsilon,N}) \cdot \mathbf{m}_t^{\varepsilon,N}\ \mathrm{d}x \\ &= \int_D \lambda_{ijkl}(x) m_j^{\varepsilon,N} \dot{m}_i^{\varepsilon,N} \epsilon_{kl}(\mathbf{n}^{\varepsilon,N})\ \mathrm{d}x \\ &= \frac{1}{2}\int_D \lambda_{ijkl}(x)(m_j^{\varepsilon,N}\dot{m}_i^{\varepsilon,N} + m_i^{\varepsilon,N}\dot{m}_j^{\varepsilon,N})\epsilon_{kl}(\mathbf{n}^{\varepsilon,N})\ \mathrm{d}x \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_D \lambda_{ijkl}(x) m_i^{\varepsilon,N} m_j^{\varepsilon,N} \epsilon_{kl}(\mathbf{n}^{\varepsilon,N}) \mathrm{d}x \\ &\quad - \frac{1}{2}\int_D \lambda_{ijkl}(x) m_i^{\varepsilon,N} m_j^{\varepsilon,N} \epsilon_{kl}(\mathbf{n}_t^{\varepsilon,N}) \mathrm{d}x, \end{aligned}$$

by symmetry of both tensor $\mathcal{S}$ and $\mathcal{L}$, we obtain

$$\begin{aligned} &\int_D \Big(\mathcal{S}(\mathbf{n}^{\varepsilon,N}) + \frac{1}{2}\mathcal{L}(\mathbf{m}^{\varepsilon,N})\Big) \cdot \epsilon(\mathbf{n}_t^{\varepsilon,N})\ \mathrm{d}x \\ &= \int_D \Big(\mathcal{S}(\mathbf{n}^{\varepsilon,N}) + \frac{1}{2}\mathcal{L}(\mathbf{m}^{\varepsilon,N})\Big) \cdot \nabla\mathbf{n}_t^{\varepsilon,N}\ \mathrm{d}x, \end{aligned}$$

and

$$\int_D \sigma_{ijkl}(x)\epsilon_{ij}(\mathbf{n}^{\varepsilon,N})\epsilon_{kl}(\mathbf{n}_t^{\varepsilon,N})\ \mathrm{d}x$$

$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_D \sigma_{ijkl}(x)\epsilon_{ij}(\mathbf{n}^{\varepsilon,N})\epsilon_{kl}(\mathbf{n}^{\varepsilon,N})\ \mathrm{d}x.$$

By using the Lemma $IV.4$, we have

$$\begin{cases} \int_D |\mathbf{m}_t^{\varepsilon,N}|^2\ \mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_D a(x)\mu(x)|\Lambda^\alpha \mathbf{m}^{\varepsilon,N}|^2\ \mathrm{d}x \\ +\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_D \mu(x)\lambda_{ijkl}(x)m_i^{\varepsilon,N}m_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{n}^{\varepsilon,N})\ \mathrm{d}x \\ -\frac{1}{2}\int_D \mu(x)\lambda_{ijkl}(x)m_i^{\varepsilon,N}m_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{n}_t^{\varepsilon,N})\ \mathrm{d}x \\ +\frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{dt}}\int_D (|\mathbf{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x = 0 \\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_D \rho(x)|\mathbf{n}_t^{\varepsilon,N}|^2\ \mathrm{d}x \\ +\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_D \sigma_{ijkl}(x)\epsilon_{ij}(\mathbf{n}^{\varepsilon,N})\epsilon_{kl}(\mathbf{n}^{\varepsilon,N})\ \mathrm{d}x \\ +\frac{1}{2}\int_D \lambda_{ijkl}(x)m_i^{\varepsilon,N}m_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{n}_t^{\varepsilon,N})\ \mathrm{d}x \\ +\int_D \mathbf{h}^N\cdot\mathbf{n}_t^{\varepsilon,N}\mathrm{d}x = 0 \end{cases}$$

By summing the both equations, we obtain

$$\int_D |\mathbf{m}_t^{\varepsilon,N}|^2\ \mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_D a(x)|\Lambda^\alpha \mathbf{m}^{\varepsilon,N}|^2\ \mathrm{d}x$$
$$+\frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{dt}}\int_D (|\mathbf{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_D \rho(x)|\mathbf{n}_t^{\varepsilon,N}|^2\ \mathrm{d}x$$
$$+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_D \sigma_{ijkl}(x)\epsilon_{ij}(\mathbf{n}^{\varepsilon,N})\epsilon_{kl}(\mathbf{n}^{\varepsilon,N})\ \mathrm{d}x$$
$$+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_D \lambda_{ijkl}(x)m_i^{\varepsilon,N}m_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{n}^{\varepsilon,N})\ \mathrm{d}x + \int_D \mathbf{h}^N\cdot\mathbf{n}_t^{\varepsilon,N}\mathrm{d}x$$
$$+\frac{1}{2}\int_D (1-\mu(x))\lambda_{ijkl}(x)m_i^{\varepsilon,N}m_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{n}_t^{\varepsilon,N})\ \mathrm{d}x = 0.$$

Now integrating in time

$$\begin{aligned} &\int_Q |\mathbf{m}_t^{\varepsilon,N}|^2\ \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_D a(x)|\Lambda^\alpha \mathbf{m}^{\varepsilon,N}(T)|^2\ \mathrm{d}x \\ &+\frac{1}{4\varepsilon}\int_D (|\mathbf{m}^{\varepsilon,N}(T)|^2-1)^2\ \mathrm{d}x + \frac{1}{2}\int_D \rho(x)|\mathbf{n}_t^{\varepsilon,N}(T)|^2\ \mathrm{d}x \\ &+\frac{1}{2}\int_D \sigma_{ijkl}(x)\epsilon_{ij}(\mathbf{n}^{\varepsilon,N})\epsilon_{kl}(\mathbf{n}^{\varepsilon,N})(T)\ \mathrm{d}x \\ &+\frac{1}{2}\int_D \lambda_{ijkl}(x)m_i^{\varepsilon,N}m_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{n}^{\varepsilon,N})(T)\ \mathrm{d}x \\ &+\frac{1}{2}\int_Q (1-\mu(x))\lambda_{ijkl}(x)m_i^{\varepsilon,N}m_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{n}_t^{\varepsilon,N})\ \mathrm{d}x\mathrm{d}t \\ &= -\int_Q \mathbf{h}^N\cdot\mathbf{n}_t^{\varepsilon,N}\mathrm{d}x + \frac{1}{2}\int_D |a(x)\Lambda^\alpha \mathbf{m}^N(0)|^2\ \mathrm{d}x \\ &+\frac{1}{4\varepsilon}\int_D (|\mathbf{m}^N(0)|^2-1)^2\ \mathrm{d}x + \frac{1}{2}\int_D \rho(x)|\mathbf{n}_t^N(0)|^2\ \mathrm{d}x \\ &+\frac{1}{2}\int_D \sigma_{ijkl}(x)\epsilon_{ij}(\mathbf{n}^N)\epsilon_{kl}(\mathbf{n}^N)(0)\ \mathrm{d}x \\ &+\frac{1}{2}\int_D \lambda_{ijkl}(x)m_i^N m_j^N\epsilon_{kl}(\mathbf{n}^N)(0)\ \mathrm{d}x. \end{aligned} \tag{13}$$

We call $\mathcal{B}^{\varepsilon,N}(T)$ the left hand side of (13) and $\mathcal{B}^N(0)$ its right hand side.

Now for a positive parameter λ such that $\frac{2\lambda}{9} > \sup_{ijkl}|\lambda_{ijkl}(x)|$ we have by Young's inequality, omitting superscripts,

$$\begin{aligned} |\lambda_{ijkl}(x)m_im_j\epsilon_{kl}(\mathbf{n}_t)| &\le \frac{2\lambda}{9}|m_i||m_j||\epsilon_{kl}(\mathbf{n}_t)| \\ &\le \frac{2\lambda}{9}\Big(\frac{\lambda}{\beta}|m_i|^2|m_j|^2 + \frac{\beta}{4\lambda}|\epsilon_{kl}(\mathbf{n}_t)|^2\Big). \end{aligned}$$

From where

$$\begin{aligned} &\sum_{ijkl}|\lambda_{ijkl}(x)m_im_j\epsilon_{kl}(\mathbf{n}_t)| \\ &\le \frac{2\lambda}{9}\Big(\frac{9\lambda}{\beta}\sum_i|m_i|^2\sum_j|m_j|^2 + \frac{9\beta}{4\lambda}\sum_{kl}|\epsilon_{kl}(\mathbf{n}_t)|^2\Big) \\ &= 2\lambda\Big(\frac{\lambda}{\beta}\big(\sum_i|m_i|^2\big)^2 + \frac{\beta}{4\lambda}\sum_{kl}|\epsilon_{kl}(\mathbf{n}_t)|^2\Big) \\ &= \frac{2\lambda^2}{\beta}|\mathbf{m}|^4 + \frac{\beta}{2}\sum_{kl}|\epsilon_{kl}(\mathbf{n}_t)|^2. \end{aligned}$$

Inspired by the work of Valente [23], we have

$$\begin{aligned} &\frac{1}{2}\Big|\int_Q (1-\mu(x))\lambda_{ijkl}(x)m_im_j\epsilon_{kl}(\mathbf{n}_t)\mathrm{d}x\mathrm{d}t\Big| \\ &= \frac{1}{2}\Big|\int_Q (1-\mu(x))\sum_{ijkl}\lambda_{ijkl}(x)m_im_j\epsilon_{kl}(\mathbf{n}_t)\mathrm{d}x\mathrm{d}t\Big| \\ &\le \frac{1}{2}\int_Q\sum_{ijkl}|\lambda_{ijkl}(x)m_im_j\epsilon_{kl}(\mathbf{n}_t)|\mathrm{d}x\mathrm{d}t \\ &\le \frac{\lambda^2}{\beta}\int_Q|\mathbf{m}|^4\mathrm{d}x\mathrm{d}t + \frac{\beta}{4}\int_Q\sum_{kl}|\epsilon_{kl}(\mathbf{n}_t)|^2\mathrm{d}x\mathrm{d}t \\ &= \frac{\lambda^2}{\beta}\int_Q\Big(|\mathbf{m}|^2-1+1\Big)^2\mathrm{d}x\mathrm{d}t \\ &\quad+\frac{\beta}{4}\int_Q\sum_{kl}|\epsilon_{kl}(\mathbf{n}_t)|^2\mathrm{d}x\mathrm{d}t \\ &\le \frac{2\lambda^2}{\beta}\int_Q\Big(|\mathbf{m}|^2-1\Big)^2\mathrm{d}x\mathrm{d}t + \frac{2\lambda^2T}{\beta}vol(D) \\ &\quad+\frac{\beta}{4}\int_Q\sum_{kl}|\epsilon_{kl}(\mathbf{n}_t)|^2\mathrm{d}x\mathrm{d}t \\ &\le \frac{2\lambda^2}{\beta}\int_Q\Big(|\mathbf{m}|^2-1\Big)^2\mathrm{d}x\mathrm{d}t + \frac{2\lambda^2T}{\beta}vol(Q) \\ &\quad+\frac{1}{4}\int_Q\sigma_{ijkl}(x)\epsilon_{ij}(\mathbf{n})\epsilon_{kl}(\mathbf{n}_t)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

by using (4). Now, for $\varepsilon < \frac{\beta}{16\lambda^2}$ we have

$$\begin{aligned} &\frac{1}{2}\Big|\int_Q (1-\mu(x))\lambda_{ijkl}(x))m_im_j\epsilon_{kl}(\mathbf{n})\mathrm{d}x\mathrm{d}t\Big| \\ &\le \frac{1}{8\varepsilon}\int_Q\Big(|\mathbf{m}|^2-1\Big)^2\mathrm{d}x\mathrm{d}t + \frac{2\lambda^2T}{\beta}vol(D) \\ &\quad+\frac{1}{4}\int_Q\sigma_{ijkl}(x))\epsilon_{ij}(\mathbf{n}_t)\epsilon_{kl}(\mathbf{n})\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Which implies

$$-\frac{1}{8\varepsilon}\int_Q\left(|\mathbf{m}^{\varepsilon,N}|^2-1\right)^2\mathrm{d}x\mathrm{d}t-\frac{2\lambda^2T}{\beta}vol(D)$$
$$-\frac{1}{4}\int_Q\sigma_{ijkl}(x))\epsilon_{ij}(\mathbf{n}^{\varepsilon,N})\epsilon_{kl}(\mathbf{n}_t^{\varepsilon,N})\mathrm{d}x\mathrm{d}t$$
$$\leq\frac{1}{2}\int_Q(1-\mu(x))\lambda_{ijkl}(x))m_i^{\varepsilon,N}m_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{n}_t^{\varepsilon,N})\mathrm{d}x\mathrm{d}t.$$

also, we have

$$\frac{1}{2}\int_D\lambda_{ijkl}(x))m_i^Nm_j^N\epsilon_{kl}(\mathbf{n}^N)(0)\mathrm{d}x$$
$$\leq\frac{1}{8\varepsilon}\int_D\left(|\mathbf{m}^N(0)|^2-1\right)^2\mathrm{d}x+\frac{2\lambda^2}{\beta}vol(D)$$
$$+\frac{1}{4}\int_D\sigma_{ijkl}(x))\epsilon_{ij}(\mathbf{n}^N)\epsilon_{kl}(\mathbf{n}^N)(0)\mathrm{d}x,$$

and

$$-\frac{1}{8\varepsilon}\int_D\left(|\mathbf{m}^{\varepsilon,N}(T)|^2-1\right)^2\mathrm{d}x-\frac{2\lambda^2}{\beta}vol(D)$$
$$-\frac{1}{4}\int_D\sigma_{ijkl}(x))\epsilon_{ij}(\mathbf{n}^{\varepsilon,N})\epsilon_{kl}(\mathbf{n}^{\varepsilon,N})(T)\mathrm{d}x$$
$$\leq\frac{1}{2}\int_D\lambda_{ijkl}(x))m_i^{\varepsilon,N}m_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{n}^{\varepsilon,N})(T)\mathrm{d}x.$$

According to the definition of $\mathcal{B}^{\varepsilon,N}(T)$ and $\mathcal{B}^N(0)$ we can write

$$\int_Q|\mathbf{m}_t^{\varepsilon,N}|^2\ \mathrm{d}x\mathrm{d}t+\frac{a_0}{2}\int_D|\Lambda^\alpha\mathbf{m}^{\varepsilon,N}(T)|^2\ \mathrm{d}x$$
$$+\tfrac{1}{8\varepsilon}\int_D(|\mathbf{m}^{\varepsilon,N}(T)|^2-1)^2\ \mathrm{d}x+\frac{\rho_0}{2}\int_D|\mathbf{n}_t^{\varepsilon,N}(T)|^2\ \mathrm{d}x$$
$$+\tfrac{1}{4}\int_D\sigma_{ijkl}(x))\epsilon_{ij}(\mathbf{n}^{\varepsilon,N})\epsilon_{kl}(\mathbf{n}^{\varepsilon,N})(T)\ \mathrm{d}x$$
$$-\tfrac{2\lambda^2(T+1)}{\beta}vol(D)-\tfrac{1}{8\varepsilon}\int_Q\left(|\mathbf{m}^{\varepsilon,N}|^2-1\right)^2\mathrm{d}x\mathrm{d}t$$
$$-\frac{1}{4}\int_Q\sigma_{ijkl}(x))\epsilon_{ij}(\mathbf{n}^{\varepsilon,N})\epsilon_{kl}(\mathbf{n}_t^{\varepsilon,N})\mathrm{d}x\mathrm{d}t\leq\mathcal{B}^{\varepsilon,N}(T),$$

and

$$\mathcal{B}^N(0)\leq-\int_Q\mathbf{h}^N\cdot\mathbf{n}_t^{\varepsilon,N}\mathrm{d}x+\frac{a_1}{2}\int_D|\Lambda^\alpha\mathbf{m}^N(0)|^2\ \mathrm{d}x$$
$$+\frac{3}{8\varepsilon}\int_D(|\mathbf{m}^N(0)|^2-1)^2\ \mathrm{d}x\ +\frac{\rho_1}{2}\int_D|\mathbf{n}_t^N(0)|^2\ \mathrm{d}x$$
$$+\frac{3}{4}\int_D\sigma_{ijkl}(x))\epsilon_{ij}(\mathbf{n}^N)\epsilon_{kl}(\mathbf{n}^N)(0)\ \mathrm{d}x+\frac{2\lambda^2}{\beta}vol(D).$$

Since $\mathcal{B}^{\varepsilon,N}(T)=\mathcal{B}^N(0)$, we have

$$\int_Q|\mathbf{m}_t^{\varepsilon,N}|^2\ \mathrm{d}x\mathrm{d}t+\frac{a_0}{2}\int_D|\Lambda^\alpha\mathbf{m}^{\varepsilon,N}(T)|^2\ \mathrm{d}x$$
$$+\frac{1}{8\varepsilon}\int_D(|\mathbf{m}^{\varepsilon,N}(T)|^2-1)^2\ \mathrm{d}x+\frac{\rho_0}{2}\int_D|\mathbf{n}_t^{\varepsilon,N}(T)|^2\ \mathrm{d}x$$
$$+\frac{1}{4}\int_D\sigma_{ijkl}(x))\epsilon_{ij}(\mathbf{n}^{\varepsilon,N})\epsilon_{kl}(\mathbf{n}^{\varepsilon,N})(T)\ \mathrm{d}x\beta vol(D)$$
$$-\frac{2\lambda^2T}{\beta}vol(D)-\frac{1}{8\varepsilon}\int_Q\left(|\mathbf{m}^{\varepsilon,N}|^2-1\right)^2\mathrm{d}x\mathrm{d}t$$
$$-\frac{1}{4}\int_Q\sigma_{ijkl}(x))\epsilon_{ij}(\mathbf{n}^{\varepsilon,N})\epsilon_{kl}(\mathbf{n}_t^{\varepsilon,N})\mathrm{d}x\mathrm{d}t$$
$$\leq-\int_Q\mathbf{h}^N\cdot\mathbf{n}_t^{\varepsilon,N}\mathrm{d}x+\frac{a_1}{2}\int_D|\Lambda^\alpha\mathbf{m}^N(0)|^2\ \mathrm{d}x$$
$$+\frac{3}{8\varepsilon}\int_D(|\mathbf{m}^N(0)|^2-1)^2\ \mathrm{d}x+\frac{\rho_1}{2}\int_D|\mathbf{n}_t^N(0)|^2\ \mathrm{d}x$$
$$+\frac{3}{4}\int_D\sigma_{ijkl}(x))\epsilon_{ij}(\mathbf{n}^N)\epsilon_{kl}(\mathbf{n}^N)(0)\ \mathrm{d}x+\frac{4\lambda^2}{\beta}vol(D).$$

At this point, we introduce the functional:

$$\mathcal{F}^{\varepsilon,N}(T)$$
$$=\int_Q|\mathbf{m}_t^{\varepsilon,N}|^2\ \mathrm{d}x\mathrm{d}t+\frac{a_0}{2}\int_D|\Lambda^\alpha\mathbf{m}^{\varepsilon,N}(T)|^2\ \mathrm{d}x$$
$$+\tfrac{1}{8\varepsilon}\int_D(|\mathbf{m}^{\varepsilon,N}(T)|^2-1)^2\ \mathrm{d}x$$
$$-\tfrac{1}{8\varepsilon}\int_Q\left(|\mathbf{m}^{\varepsilon,N}|^2-1\right)^2\mathrm{d}x\mathrm{d}t+\frac{\rho_0}{2}\int_D|\mathbf{n}_t^{\varepsilon,N}(T)|^2\ \mathrm{d}x$$
$$+\tfrac{\beta}{4}\int_D|\nabla\mathbf{n}^{\varepsilon,N}|^2(T)\ \mathrm{d}x-\tfrac{\beta}{4}\int_D|\nabla\mathbf{n}_t^{\varepsilon,N}|^2\ \mathrm{d}x$$
$$-\tfrac{2\lambda^2T}{\beta}vol(D),$$

then

$$\mathcal{F}^{\varepsilon,N}(0)$$
$$=\frac{a_1}{2}\int_D|\Lambda^\alpha\mathbf{m}^{\varepsilon,N}(0)|^2\ \mathrm{d}x+\frac{1}{8\varepsilon}\int_D(|\mathbf{m}^{\varepsilon,N}(0)|^2-1)^2\ \mathrm{d}x$$
$$+\frac{\rho_1}{2}\int_D|\mathbf{n}_t^{\varepsilon,N}(0)|^2\ \mathrm{d}x+\frac{\beta}{4}\int_D|\nabla\mathbf{n}^{\varepsilon,N}|^2(0)\ \mathrm{d}x.$$

In addition

$$-\int_Q\mathbf{h}^N\cdot\mathbf{n}_t^{\varepsilon,N}\mathrm{d}x\leq\frac{\rho_1}{2}||u_t^{\varepsilon,N}||^2_{\mathbf{L}^2(Q)}+\frac{1}{2\rho_1}||\mathbf{h}^N||^2_{\mathbf{L}^2(Q)},$$
$$\int_D|\nabla\mathbf{n}^{\varepsilon,N}(T)|^2\ \mathrm{d}x\leq\int_D\sum_{kl}|\epsilon_{kl}(\mathbf{n}^{\varepsilon,N}(T))|^2\mathrm{d}x,$$

and under the assumption $\sigma_{ijkl}(x)\epsilon_{ij}(\mathbf{n})\epsilon_{kl}(\mathbf{n})\leq\tau|\nabla\mathbf{n}|^2$ (for a positive constant τ), we have

$$\int_Q|\mathbf{m}_t^{\varepsilon,N}|^2\ \mathrm{d}x\mathrm{d}t+\frac{a_0}{2}\int_D|\Lambda^\alpha\mathbf{m}^{\varepsilon,N}(T)|^2\ \mathrm{d}x$$
$$+\frac{1}{8\varepsilon}\int_D(|\mathbf{m}^{\varepsilon,N}(T)|^2-1)^2\ \mathrm{d}x$$
$$-\frac{1}{8\varepsilon}\int_Q\left(|\mathbf{m}^{\varepsilon,N}|^2-1\right)^2\mathrm{d}x\mathrm{d}t$$
$$-\frac{\beta}{4}\int_Q|\nabla\mathbf{n}_t^{\varepsilon,N}|^2\ \mathrm{d}x-\frac{2\lambda^2T}{\beta}vol(D)$$
$$+\frac{\rho_0}{2}\int_D|\mathbf{n}_t^{\varepsilon,N}(T)|^2\ \mathrm{d}x+\frac{\beta}{4}\int_D|\nabla\mathbf{n}^{\varepsilon,N}|^2(T)\ \mathrm{d}x\leq\quad(14)$$
$$\frac{a_1}{2}\int_D|\Lambda^\alpha\mathbf{m}^N(0)|^2\ \mathrm{d}x+\frac{3}{8\varepsilon}\int_D(|\mathbf{m}^N(0)|^2-1)^2\ \mathrm{d}x$$
$$+\frac{\rho_1}{2}\int_D|\mathbf{n}_t^N(0)|^2\ \mathrm{d}x+\frac{1}{2\rho_1}||\mathbf{h}^N||^2_{\mathbf{L}^2(Q)}$$
$$+\frac{3\tau}{4}\int_D|\nabla\mathbf{n}^N(0)|^2\ \mathrm{d}x+\frac{4\lambda^2}{\beta}vol(D),$$

which implies

$$\mathcal{F}^{\varepsilon,N}(T) \le \int_0^T \mathcal{F}^{\varepsilon,N}(t)\ \mathrm{d}t + 3\mathcal{F}^{\varepsilon,N}(0) + \frac{1}{2\rho_1}||\mathbf{h}^N||^2_{\mathbf{L}^2(Q)} + \frac{4\lambda^2}{\beta} vol(D),$$

and from the Gronwall lemma, we have

$$\mathcal{F}^{\varepsilon,N}(T) \le e^T\big(3\mathcal{F}^{\varepsilon,N}(0) + \frac{1}{2\rho_1}||\mathbf{h}^N||^2_{\mathbf{L}^2(Q)} + \frac{4\lambda^2}{\beta} vol(D)\big).$$

Since $\mathbf{n}_0 \in \mathbf{H}^1_0(D)$, $\mathbf{n}_1 \in \mathbf{L}^2(D)$ and $\mathbf{m}_0 \in \mathbf{H}^\alpha(D)$ which is embedded into $\mathbf{L}^4(D)$ for $1 < \alpha < \frac{3}{2}$ the right hand side is uniformly bounded. Indeed, for constants C_1, C_2, C_3, C_4 and $C(\mathbf{h})$ independent of N

$$\begin{aligned}
&\int_D (|\mathbf{m}^N(0)|^2 - 1)^2\ \mathrm{d}x \\
&= \int_D |\mathbf{m}^N(0)|^4\ \mathrm{d}x - 2\int_D |\mathbf{m}^N(0)|^2\ \mathrm{d}x + vol(D) \\
&\le ||\mathbf{m}^N(0)||^4_{\mathbf{L}^4(D)} + vol(D) \\
&\le C_1||\mathbf{m}^N(0)||^4_{\mathbf{H}^\alpha(D)} + C_2 \\
&\le C_3,
\end{aligned}$$

$$\begin{aligned}
&\int_D |\nabla \mathbf{n}^N(0)|^2\ \mathrm{d}x \\
&= \int_D |\nabla \mathbf{n}^N(0) - \nabla \mathbf{n}_0 + \nabla \mathbf{n}_0|^2\ \mathrm{d}x \\
&\le 2\int_D |\nabla \mathbf{n}^N(0) - \nabla \mathbf{n}_0|^2\ \mathrm{d}x + 2\int_D |\nabla \mathbf{n}_0|^2\ \mathrm{d}x \\
&\le 2\int_D |\nabla \mathbf{n}^N(0) - \nabla \mathbf{n}_0|^2\ \mathrm{d}x + 2\int_D |\nabla \mathbf{n}_0|^2\ \mathrm{d}x \\
&\le 2||\mathbf{n}^N(0) - \mathbf{n}_0||^2_{\mathbf{H}^1_0(D)} + 2||\mathbf{n}_0||^2_{\mathbf{H}^1_0(D)} \\
&\le C_4,
\end{aligned}$$

and

$$\begin{aligned}
||\mathbf{h}^N||^2_{\mathbf{L}^2(Q)} &= ||\mathbf{h}^N - \mathbf{h} + \mathbf{h}||^2_{\mathbf{L}^2(Q)} \\
&\le 2||\mathbf{h}^N - \mathbf{h}||^2_{\mathbf{L}^2(Q)} + 2||\mathbf{h}||^2_{\mathbf{L}^2(Q)} \\
&\le C(\mathbf{h}),
\end{aligned}$$

Due to the strong convergences $\mathbf{m}^N(.,0) \to \mathbf{m}_0$ in $\mathbf{H}^\alpha(D)$, $\mathbf{n}^N(.,0) \to \mathbf{n}_0$ in $\mathbf{H}^1_0(D)$ and $\mathbf{h}^N(x) \to \mathbf{h}(x)$ in $\mathbf{L}^2(Q)$. For the other term $(\mathbf{n}^N_t(0))$, the estimate can be carried out in an analogous way using the strong convergence $\mathbf{n}^N_t(.,0) \to \mathbf{n}_1$ in $\mathbf{L}^2(D)$. Moreover, noting that (for a constant C independent of ε and N)

$$\begin{aligned}
\int_D |\mathbf{m}^{\varepsilon,N}|^2\ \mathrm{d}x &= \int_D (|\mathbf{m}^{\varepsilon,N}|^2 - 1 + 1)\ \mathrm{d}x \\
&\le \frac{1}{2}\int_D (|\mathbf{m}^{\varepsilon,N}|^2 - 1)^2\ \mathrm{d}x + C.
\end{aligned}$$

Therefore, for a fixed parameter $\varepsilon > 0$ we have

$$\begin{aligned}
&(\mathbf{m}^{\varepsilon,N})_N \ \text{ is bounded in } \ L^\infty(0,T;\mathbf{H}^\alpha(D)), \\
&(\mathbf{m}^{\varepsilon,N}_t)_N \ \text{ is bounded in } \ L^2(0,T;\mathbf{L}^2(D)), \\
&(|\mathbf{m}^{\varepsilon,N}|^2 - 1)_N \ \text{ is bounded in } \ L^\infty(0,T;L^2(D)), \\
&(\mathbf{n}^{\varepsilon,N})_N \ \text{ is bounded in } \ L^2(0,T;\mathbf{H}^1_0(D)), \qquad (15) \\
&(\mathbf{n}^{\varepsilon,N}_t)_N \ \text{ is bounded in } \ L^2(0,T;\mathbf{L}^2(D)).
\end{aligned}$$

Note that, (15) is due to the Poincaré lemma. classical compactness results imply the existence of two subsequences still denoted by $(\mathbf{m}^{\varepsilon,N})$ and $(\mathbf{n}^{\varepsilon,N})$ such that for fixed $\varepsilon > 0$

$$\begin{aligned}
&\mathbf{m}^{\varepsilon,N} \rightharpoonup \mathbf{m}^\varepsilon \quad \text{weakly in} \quad L^2(0,T;\mathbf{H}^\alpha(D)), \\
&\mathbf{m}^{\varepsilon,N}_t \rightharpoonup \mathbf{m}^\varepsilon_t \quad \text{weakly in} \quad \mathbf{L}^2(Q), \\
&\mathbf{m}^{\varepsilon,N} \to \mathbf{m}^\varepsilon \ \text{ strongly in } \ L^2(0,T,\mathbf{H}^\beta(D)) \qquad (16) \\
&\text{and a.e. for } \ 0 \le \beta < \alpha \\
&|\mathbf{m}^{\varepsilon,N}|^2 - 1 \rightharpoonup \zeta \quad \text{weakly in} \quad L^2(Q), \\
&\mathbf{n}^{\varepsilon,N} \rightharpoonup \mathbf{n}^\varepsilon \quad \text{weakly in} \quad L^2(0,T;\mathbf{H}^1_0(D)), \\
&\mathbf{n}^{\varepsilon,N}_t \rightharpoonup \mathbf{n}^\varepsilon_t \quad \text{weakly in} \quad \mathbf{L}^2(Q), \\
&\mathbf{n}^{\varepsilon,N} \to \mathbf{n}^\varepsilon \quad \text{strongly in} \quad \mathbf{L}^2(Q).
\end{aligned}$$

The convergence (16) is due to Lemma $IV.1$ and thanks to Lemma $IV.2$ it can be shown that $\zeta = |\mathbf{m}^\varepsilon|^2 - 1$. based on the Sobolev embedding $H^\alpha(Q) \hookrightarrow L^4(Q)$, since $1 < \alpha < \frac{3}{2}$,the subsequent compactness result ensues

$$m^{\varepsilon,N}_i m^{\varepsilon,N}_j \to m^\varepsilon_i m^\varepsilon_j \ \text{ strongly in } \ L^2(Q), \qquad (17)$$

and

$$m^{\varepsilon,N}_i \phi_j \to m^\varepsilon_i \phi_j \ \text{ strongly in } \ L^2(Q).$$

Therefore, we consider the variational formulation of (12).

$$\begin{cases}
\int_Q \mathbf{m}^{\varepsilon,N}_t \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t + \int_Q a(x)\mu(x)\Lambda^\alpha \mathbf{m}^{\varepsilon,N} \cdot \Lambda^\alpha \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t \\
-\int_Q \mu(x)\lambda_{ijkl}(x) m^{\varepsilon,N}_i m^{\varepsilon,N}_j \epsilon_{kl}(\mathbf{n}^{\varepsilon,N}_t)\boldsymbol{\phi}_i\ \mathrm{d}x\mathrm{d}t \\
+\int_Q \mu(x)\lambda_{ijkl} m^{\varepsilon,N}_i m^{\varepsilon,N}_j \epsilon_{kl}(\mathbf{n}^{\varepsilon,N})\boldsymbol{\phi}_i\ \mathrm{d}x\mathrm{d}t \\
+\int_Q \frac{|\mathbf{m}^{\varepsilon,N}|^2 - 1}{\varepsilon}\mathbf{m}^{\varepsilon,N} \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t = 0 \\
-\ \int_Q \rho(x)\mathbf{n}^{\varepsilon,N}_t \cdot \boldsymbol{\psi}_t\ \mathrm{d}x\mathrm{d}t \\
+\int_Q \sigma_{ijkl}(x)\epsilon_{ij}(\mathbf{n}^{\varepsilon,N})\epsilon_{kl}(\boldsymbol{\psi})\ \mathrm{d}x\mathrm{d}t \\
+\frac{1}{2}\int_Q \lambda_{ijkl}(x) m^{\varepsilon,N}_i m^{\varepsilon,N}_j \epsilon_{kl}(\boldsymbol{\psi})\ \mathrm{d}x\mathrm{d}t \\
+\int_Q \mathbf{h}^N \cdot \boldsymbol{\psi}\ \mathrm{d}x\mathrm{d}t = 0,
\end{cases} \qquad (18)$$

for any $\boldsymbol{\phi} \in L^2(0,T;\mathbf{H}^\alpha(D))$ and $\boldsymbol{\psi} \in \mathbf{H}^1_0(Q)$.
Taking the limit $N \to \infty$ in (18), we obtain

$$\begin{cases} \int_Q \mathbf{m}_t^\varepsilon \cdot \boldsymbol{\phi} \ \mathrm{d}x\mathrm{d}t + \int_Q a(x)\mu(x)\Lambda^\alpha \mathbf{m}^\varepsilon \cdot \Lambda^\alpha \boldsymbol{\phi} \ \mathrm{d}x\mathrm{d}t \\ - \int_Q \mu(x)\lambda_{ijkl}(x) m_i^{\varepsilon,N} m_j^{\varepsilon,N} \epsilon_{kl}(\mathbf{n}_t^{\varepsilon,N}) \boldsymbol{\phi}_i \ \mathrm{d}x\mathrm{d}t \\ + \int_Q \mu(x)\lambda_{ijkl} m_i^\varepsilon m_j^\varepsilon \epsilon_{kl}(\mathbf{n}^\varepsilon)\boldsymbol{\phi}_i \ \mathrm{d}x\mathrm{d}t \\ + \int_Q \frac{|\mathbf{m}^\varepsilon|^2 - 1}{\varepsilon} \mathbf{m}^\varepsilon \cdot \boldsymbol{\phi} \ \mathrm{d}x\mathrm{d}t = 0 \\ - \int_Q \rho(x) \mathbf{n}_t^\varepsilon \cdot \boldsymbol{\psi}_t \ \mathrm{d}x\mathrm{d}t + \int_Q \sigma_{ijkl}(x)\epsilon_{ij}(\mathbf{n}^\varepsilon)\epsilon_{kl}(\boldsymbol{\psi}) \ \mathrm{d}x\mathrm{d}t \\ + \frac{1}{2}\int_Q \lambda_{ijkl}(x) m_i^\varepsilon m_j^\varepsilon \epsilon_{kl}(\boldsymbol{\psi}) \ \mathrm{d}x\mathrm{d}t + \int_Q \mathbf{h}\cdot\boldsymbol{\psi} \ \mathrm{d}x\mathrm{d}t = 0, \end{cases} \tag{19}$$

for any $\boldsymbol{\phi} \in L^2(0,T;\mathbf{H}^\alpha(D))$ and $\boldsymbol{\psi} \in \mathbf{H}_0^1(Q)$. We proved the following result.

Proposition V.1. *Let $\mathbf{m}_0 \in \mathbf{H}^\alpha(D)$ such that $|\mathbf{m}_0| = 1$ a.e., $\mathbf{n}_0 \in \mathbf{H}_0^1(D)$ and $\mathbf{n}_1 \in \mathbf{L}^2(D)$. Then, for any positive ε small enough and any fixed time T, There exists a solution $\mathbf{m}^\varepsilon$, to the problem (11) in the sense of distributions. Moreover, we have the following energy estimate*

$$\begin{aligned} &\int_Q |\mathbf{m}_t^\varepsilon|^2 \ \mathrm{d}x\mathrm{d}t + \frac{a_0}{2}\int_D |\Lambda^\alpha \mathbf{m}^\varepsilon(T)|^2 \ \mathrm{d}x \\ &+\frac{1}{8\varepsilon}\int_D (|\mathbf{m}^\varepsilon(T)|^2 - 1)^2 \ \mathrm{d}x + \frac{\rho_0}{2}\int_D |\mathbf{n}_t^\varepsilon(T)|^2 \ \mathrm{d}x \\ &+\frac{\beta}{4}\int_D |\nabla \mathbf{n}^\varepsilon|^2(T) \ \mathrm{d}x - \frac{\beta}{4}\int_Q |\nabla \boldsymbol{n}_t^{\varepsilon,N}|^2 \ \mathrm{d}x\mathrm{d}t \\ &-\frac{2\lambda^2 T}{\beta} vol(D) \le \frac{a_1}{2}\int_D |\Lambda^\alpha \mathbf{m}_0|^2 \ \mathrm{d}x \\ &+\frac{\rho_1}{2}\int_D |\mathbf{n}_1|^2 \ \mathrm{d}x + \frac{3\tau}{4}\int_D |\nabla \mathbf{n}_0|^2 \ \mathrm{d}x \\ &+\frac{4\lambda^2}{\beta} vol(D) + C(\mathbf{h}). \end{aligned} \tag{20}$$

Remark V.2. *By taking the lower semicontinuous limit in* (14)*,We can deduce* (20)

B. Convergence of approximate solutions

Our aim here is to take the limit as $\varepsilon \to 0$. Based on the estimate given in (20), we can conclude that the following quantities are uniformly bounded.

$$\begin{aligned} &(\mathbf{m}^\varepsilon)_\varepsilon \ \text{ is bounded in } \ L^\infty(0,T;\mathbf{H}^\alpha(D)), \\ &(\mathbf{m}_t^\varepsilon)_\varepsilon \ \text{ is bounded in } \ L^2(0,T;\mathbf{L}^2(D)), \\ &(|\mathbf{m}^\varepsilon|^2 - 1)_\varepsilon \ \text{ is bounded in } \ L^\infty(0,T;L^2(D)), \\ &(\mathbf{n}^\varepsilon)_\varepsilon \ \text{ is bounded in } \ L^2(0,T;\mathbf{H}_0^1(D)), \\ &(\mathbf{n}_t^\varepsilon)_\varepsilon \ \text{ is bounded in } \ L^2(0,T;\mathbf{L}^2(D)). \end{aligned}$$

Subsequently, there are two subsequences that we continue to denote as $(\mathbf{m}^\varepsilon)$ and $(\mathbf{n}^\varepsilon)$ such that

$$\begin{aligned} &\mathbf{m}^\varepsilon \rightharpoonup \mathbf{m} \quad \text{weakly in} \quad L^2(0,T;\mathbf{H}^\alpha(D)), \\ &\mathbf{m}_t^\varepsilon \rightharpoonup \mathbf{m}_t \quad \text{weakly in} \quad L^2(0,T;\mathbf{L}^2(D)), \\ &m^\varepsilon \to \mathbf{m} \quad \text{strongly in} \quad L^2(0,T,\mathbf{H}^\beta(D)) \\ &\qquad \text{and a.e. for } \ 0 \le \beta < \alpha \\ &|\mathbf{m}^\varepsilon|^2 - 1 \to 0 \quad \text{strongly in } L^2(Q) \ \text{ and a.e.} \\ &\mathbf{n}^\varepsilon \rightharpoonup \mathbf{n} \quad \text{weakly in} \quad L^2(0,T;\mathbf{H}_0^1(D)), \\ &\mathbf{n}_t^\varepsilon \rightharpoonup \mathbf{n}_t \quad \text{weakly in} \quad \mathbf{L}^2(Q), \\ &\mathbf{n}^\varepsilon \to \mathbf{n} \quad \text{strongly in} \quad \mathbf{L}^2(Q). \end{aligned} \tag{21}$$

We can deduce that $|\mathbf{m}| = 1$ a.e.,by using the convergence (21)

To take the limit as ε approaches 0 in equation (19), let $\boldsymbol{\phi} = \mathbf{m}^\varepsilon \times \boldsymbol{\varphi}$ where $\boldsymbol{\varphi} \in \mathbf{C}^\infty(\overline{Q})$. Since $\boldsymbol{\phi}$ belongs to $L^2(0,T;\mathbf{H}^\alpha(D))$, the following holds:

$$\begin{cases} \int_Q \mathbf{m}_t^\varepsilon \cdot (\mathbf{m}^\varepsilon \times \boldsymbol{\varphi}) \ \mathrm{d}x\mathrm{d}t \\ + \int_Q a(x)\mu(x)\Lambda^\alpha \mathbf{m}^\varepsilon \cdot \Lambda^\alpha (\mathbf{m}^\varepsilon \times \boldsymbol{\varphi}) \ \mathrm{d}x\mathrm{d}t \\ - \int_Q \mu(x)\lambda_{ijkl}(x) m_i^\varepsilon m_j^\varepsilon \epsilon_{kl}(\mathbf{n}_t^\varepsilon)\boldsymbol{\phi}_i \ \mathrm{d}x\mathrm{d}t \\ + \int_Q \mu(x)\lambda_{ijkl}(x) m_j^\varepsilon \epsilon_{kl}(\mathbf{n}^\varepsilon)(\mathbf{m}^\varepsilon \times \boldsymbol{\varphi})_i \ \mathrm{d}x\mathrm{d}t = 0 \\ - \int_Q \rho(x)\mathbf{n}_t^\varepsilon \cdot \boldsymbol{\psi}_t \ \mathrm{d}x\mathrm{d}t + \int_Q \sigma_{ijkl}(x)\epsilon_{ij}(\mathbf{n}^\varepsilon)\epsilon_{kl}(\boldsymbol{\psi}) \ \mathrm{d}x\mathrm{d}t \\ + \frac{1}{2}\int_Q \lambda_{ijkl}(x) m_i^\varepsilon m_j^\varepsilon \epsilon_{kl}(\boldsymbol{\psi}) \ \mathrm{d}x\mathrm{d}t + \int_Q \mathbf{h}\cdot\boldsymbol{\psi} \ \mathrm{d}x\mathrm{d}t = 0. \end{cases} \tag{22}$$

Due to recent convergences we have established, and a result similar to the one in equation (17), and based on ([7]) We take the limit in (22) when $\varepsilon \to 0$

Hence

$$\begin{cases} \int_Q \mathbf{m}_t \cdot (\mathbf{m} \times \boldsymbol{\varphi}) \ \mathrm{d}x\mathrm{d}t \\ + \int_Q a(x)\mu(x)\Lambda^\alpha \mathbf{m} \cdot \Lambda^\alpha (\mathbf{m} \times \boldsymbol{\varphi}) \ \mathrm{d}x\mathrm{d}t \\ - \int_Q \mu(x)\lambda_{ijkl}(x) m_i m_j \epsilon_{kl}(\mathbf{n}_t)(\mathbf{m} \times \boldsymbol{\varphi})_i \ \mathrm{d}x\mathrm{d}t \\ + \int_Q \mu(x)\lambda_{ijkl}(x) m_i m_j \epsilon_{kl}(\mathbf{n})(\mathbf{m} \times \boldsymbol{\varphi})_i \ \mathrm{d}x\mathrm{d}t = 0 \\ - \int_Q \rho(x)\mathbf{n}_t \cdot \boldsymbol{\psi}_t \ \mathrm{d}x\mathrm{d}t + \int_Q \sigma_{ijkl}\epsilon_{ij}(\mathbf{n})\epsilon_{kl}(\boldsymbol{\psi}) \ \mathrm{d}x\mathrm{d}t \\ + \frac{1}{2}\int_Q \lambda_{ijkl} m_i m_j \epsilon_{kl}(\boldsymbol{\psi}) \ \mathrm{d}x\mathrm{d}t + \int_Q \mathbf{h}\cdot\boldsymbol{\psi} \ \mathrm{d}x\mathrm{d}t = 0, \end{cases}$$

for all $\boldsymbol{\varphi} \in \mathbf{C}^\infty(\overline{Q})$ and $\boldsymbol{\psi} \in \mathbf{H}_0^1(Q)$. It is worth noting that, from the estimate given in equation (20), one can readily obtain equation (10).Therefore, $(\mathbf{m}, \mathbf{n})$ is a solution to the problem (5)-(6)-(7) in the sense of the definition in $III.1$, thus completing the proof of Theorem $III.2$.

Remark V.3. *If $\mu(x) = 1$, we can readily prove the existence of the global solutions of the problem defined by equations* (5)*,* (6)*, and* (7)*.*

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