Derived Subgroup and Direct Product of Groups Embedded Into Wreath Product

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Abstract: In this paper, we showed that derived subgroup and direct product groups can be embedded into wreath products of groups with examples.

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1. Introduction

Over the years, a lot of research has been done on wreath products and embedment of groups into wreath products as seen in [3,4,6,7,8], in this work we considered the case where derived subgroup and direct product are embedded into wreath products.

2. Basic Definition

Suppose that G and H are two groups, then the group $G \times H$ is also a group popularly known as the *Direct Product* of the groups G and H given by $G \times H = \{(g,h) | g \in G, h \in H\}$ and its multiplication for the group is defined to be

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2) \ldots\ldots (1)$$

If 1_G is the known identity for G, and 1_H is the known identity for H, then $(1_G, 1_H)$ is the identity for the group $G \times H$ and also $(g,h)^{-1} = (g^{-1}, h^{-1})$.

Suppose that Γ and Δ be nonempty sets, then the set of functions from the set Δ to the set Γ, it is going to be denoted by Γ^Δ. In the event that C is a group, we made C^Δ into a group also by outlining product multiplication "pointwise"

$$fg(\gamma) := f(\gamma)g(\gamma) \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

for all $f, g \in C^\Delta$ and $\gamma \in \Delta$ in which the given product on the right hand side is in the group C.

Let‟s supposed that C and D are groups and that D is a group action on the nonempty set Δ, then we define the *wreath product* of the groups C by D with respect to the given action to be the semidirect product $C^\Delta \rtimes D = CwrD$ where the group D acts on the group of functions C^Δ through

$$f^d(\gamma) := f\left(\gamma^{d^{-1}}\right) \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

for all $f \in C^{\Delta}, \gamma \in \Delta$ and, $d \in D$ and the multiplication defined for all $(f_1, d_1), (f_2, d_2) \in C\ wr\ D$ by

$(f_1, d_1)(f_2, d_2) = \left(f_1 f_2^{{d_1}^{-1}}, d_1 d_2\right)$ .....(4)

Cleary, $|C\ wrD| = |C|^{|\Delta|}|D|$……………………...(5)

Suppose we have a homomorphism $\eta: G \longrightarrow H$ that is also one-to-one (or injective), then such a homomorphism is known as an *embedding*: the group G "embeds" into H by means of a subgroup. On the condition that η is not a one-to-one mapping, then it is called a *quotient* mapping. Now if $\eta: G \longrightarrow H$ is an embedding, then it is known hat $\ker(\eta) = \{e_G\}$ and using the „First Isomorphism Theorem" of groups, $Im(\eta) \cong G/\{e_G\} \cong G$. Now also, $Im(\eta) \leq H$ since $\eta: G \longrightarrow H$ is a homomorphism, and we can assert that in any given embedding, G is always an isomorphism with a given subgroup of H.

3. Groups Embedded Into Wreath Product

Proposition 1: Let A be an abelian group, at that point the commutator (derived) subgroup of the wreath product $A\ wr\ C_2$ is embedded into the given wreath product.

Proof: Since A is a group that is abelian, then the derived subgroup $A' = \{(a, a^{-1}): a \in A\}$ of the base group A^2, of the given wreath product $A\ wr\ C_2$ is obviously isomorphic to A (See [5]). Thus embedded in $A\ wr\ C_2$, as A is isomorphic to a subgroup of $A\ wr\ C_2$.

Example: Let $A := \langle(12), (34)\rangle = \{(1), (12), (34), (12)(34)\}$ which is abelian and $C_2 := \langle(12)\rangle = \{(1), (12)\}$ then the Wreath Product

$A\ wr\ C_2 = \langle(12), (34), (56), (78), (15)(26)(37)(48)\rangle = \{(1), (78), (56), (56)(78), (34), (34)(78), (34)(56), (34)(56)(78), (12), (12)(78), (12)(56), (12)(56)(78), (12)(34), (12)(34)(78), (12)(34)(56), (12)(34)(56)(78), (15)(26)(37)(48), (15)(26)(3748), (1526)(37)(48), (1526)(3748), (15)(26)(3847), (15)(26)(38)(47), (1526)(3847), (1526)(38)(47), (1625)(37)(48), (1625)(3748), (16)(25)(37)(48), (16)(25)(3748), (1625)(3847), (1625)(38)(47), (16)(25)(3847), (16)(25)(38)(47)\}$ which is a group of order 32. Then the derived subgroup is $\langle(12)(34)(56)(78), (12)(56)\rangle = \{(1), (34)(78), (12)(56), (12)(34)(56)(78)\} \cong A$.

Proposition 2: Let A be a direct product of $p-1$ cyclic groups of order p^n, then A is embedded into the wreath product $W = C_{p^n}\ wr\ C_p$.

Proof: As A is a direct product of $p-1$ cyclic groups and $W = C_{p^n}\ wr\ C_p$, then $W' \cong A$ (See [2]). Now since $W' \trianglelefteq W$, then A is embedded in $W = C_{p^n}\ wr\ C_p$.

Example: Let $p = 3$ and $n = 2$. Then we have: $C_3 = \langle(123)\rangle = \{(1), (123), (132)\}$ and $C_{3^2} = C_9 = \langle(123456789)\rangle = \{(1), (123456789), (135792468), (147)(258)(369), (159483726), (162738495), (174)(285)(396), (186429753), (198765432)\}$ Then the Wreath Product $W = C_9\ wr\ C_3 =$

$$\left\langle \begin{matrix} (123456789), (10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18), \\ (19\ 20\ 21\ 22\ 23\ 24\ 25\ 26\ 27), \\ (1\ 10\ 19)(2\ 11\ 20)(3\ 12\ 21)(4\ 13\ 22)(5\ 14\ 23) \\ (6\ 15\ 24)(7\ 16\ 25)(8\ 17\ 26)(9\ 18\ 27) \end{matrix} \right\rangle$$

which is a group of order 2187 and the derived subgroup

$$W' = \langle (123456789)(10\ 18\ 17\ 16\ 15\ 14\ 13\ 12\ 11), (10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18)(19\ 27\ 26\ 25\ 24\ 23\ 22\ 21\ 20)\rangle$$

Which is a group of order 81 and it isomorphic to
$C_9 \times C_9 = \langle (123456789), (10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18)\rangle$
which is also a finite group of order 81.

4. Conclusion

We proved with examples how derived subgroup and direct product of groups were embedded into wreath products.

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