Guaranteeing Half-Maximin Shares Under Cardinality Constraints

Extended Abstract

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ABSTRACT

We study the problem of fair allocation of a set of indivisible items among agents with additive valuations, under cardinality constraints. In this setting, the items are partitioned into categories, each with its own limit on the number of items it may contribute to any bundle. We consider the fairness measure known as the *maximin share* (MMS) *guarantee*, and propose a novel polynomial-time algorithm for finding 1/2-approximate MMS allocations—an improvement from the previously best available guarantee of 11/30.

KEYWORDS

Constrained Fair Allocation; Indivisible Goods; Maximin Share

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1 INTRODUCTION

Fairly allocating items in the real world often involves placing constraints on what items can be allocated together. For example, one may require that agents receive bundles of connected items [3, 5, 15, 20], that items deemed conflicting are not allocated to the same agent [8, 16] or that all agents receive the same number of items [9]. Suksompong provides a recent overview of constrained fair allocation [21].

In this paper[1] we explore fair allocation under a type of constraints known as *cardinality constraints*, introduced by Biswas and Barman [4]. Under cardinality constraints, the items are partitioned into a set of *categories*. Each category has a limit on the number of items any agent may be allocated from the category. This type of constraints naturally occurs in many real-world settings such as allocating seats in a space-constrained conference with synchronized parallel tracks or, in a more general setting, preventing individual agents from receiving almost all items of a given type.

Contributions. We develop a polynomial-time algorithm for finding 1/2-approximate MMS allocations under cardinality constraints, improving on the guarantees of 1/3 and 11/30, provided by Biswas and Barman [4] and Li and Vetta [19]—to our knowledge the best guarantees previously available.

[1] A full version of the paper, with proofs and experiments, is available [17].

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Related Work. Cardinality constraints may be represented by a *partition matroid*, and there exist several works on fair allocation under matroid constraints [e.g., 13, 14]. As pointed out by Biswas and Barman [4], the 1/2-approximate MMS algorithm of Gourvès and Monnot [13] is not applicable here, as the matroid constraint is applied to the union of *all* the bundles. There is also some overlap between allocation of *conflicting items* [8, 16] and cardinality constraints, but neither is a generalization of the other.

2 PRELIMINARIES

An instance of the fair allocation problem under cardinality constraints is given by $I = \langle M, N, V, C \rangle$, where $M = \{1, \ldots, m\}$ is the set of *items*, $N = \{1, \ldots, n\}$ the set of *agents* and $V = \{v_1, \ldots, v_n\}$ a collection of *valuation functions* over subsets of M, $v_i : 2^m \to \mathbb{R}_{\geq 0}$, where v_i is the valuation function of agent i. We consider additive valuations, i.e., $v_i(S) = \sum_{j \in S} v_i(\{j\})$. Finally, C is a set of ℓ pairs $\langle C_h, k_h \rangle$ of *categories* C_h and corresponding *thresholds* k_h such that the categories constitute a partition of M. For an instance I, we wish to find a *feasible* partition of M into n, possibly empty, *bundles*. That is, we want a *complete allocation* $A = \langle A_1, A_2, \ldots, A_n \rangle$, where for all bundles A_i and categories C_h, $|A_i \cap C_h| \leq k_h$. Without loss of generality, we assume that $|C_h| \leq nk_h$ for each $h \in \{1, \ldots, \ell\}$.[2]

We are concerned with the fairness criterion of the *maximin share guarantee* [7]. The *maximin share* (MMS) of an agent is the value of the most preferred bundle the agent can guarantee herself if she were to divide the items into feasible bundles and then choose her own bundle last. Formally, for an instance $I = \langle M, N, V, C \rangle$, the *maximin share* of an agent i for the instance I, μ_i^I, is given by

$$\mu_i^I = \max_{A \in \mathcal{F}_I} \min_{A_j \in A} v_i(A_j)\,,$$

where $\mathcal{F}_I$ is the set of feasible allocations for I. If I is obvious from context, we write simply μ_i. An allocation, A, is said to be an *MMS allocation*, if $v_i(A_i) \geq \mu_i$ for all agents i. We are interested in *α-approximate MMS allocations*, which are allocations where for an $\alpha > 0$, $v_i(A_i) \geq \alpha\mu_i$ for all agents i. Our algorithm needs a way to decide if a bundle is worth at least $\alpha\mu_i$ to agent i. However, finding μ_i is known to be NP-hard [22].[3] In order to provide a polynomial-time algorithm, we exploit a common overestimate of μ_i from unconstrained fair allocation [e.g., 1, 10]. The trick is to *normalize* the instance, i.e., rescale agents' valuations so that $v_i(M) = n$. As a result, $\mu_i \leq 1$, and giving each agent a bundle valued at least α is sufficient for a α-approximate MMS allocation.

[2] If this is not the case, one can remove the least valuable items in each category after the later ordering of the instance.

[3] In the unconstrained setting, a PTAS exists for finding the MMS of an agent [22], but this PTAS does not extend to fair allocation under cardinality constraints.

3 ORDERED INSTANCES

In the unconstrained setting, Bouveret and Lemaître showed that each instance can be reduced to an instance where all agents have the same preference order over all items [6]. These instances are known as *ordered instances* [2, 10] and can be created by sorting each agent's item values and reassign these to the items in some predetermined common order. This does not change agents' MMS. Furthermore, one can convert an allocation in the ordered instance into one for the original instance such that no agent is worse off. Thus, MMS approximation need only consider ordered instances.

This approach does not work directly for cardinality constraints, as reassigning item values across categories may result in infeasibility when converting back from the ordered instance. Instead, we introduce the following modified definition, which can be shown to have similar properties as in the unconstrained setting.

Definition 1. *An instance $I = \langle M, N, V, C\rangle$ of the fair allocation problem under cardinality constraints is called an* ordered instance *if each category $C_h = \{c_1, c_2, \cdots, c_{|C_h|}\}$ is ordered such that for all agents i, $v_i(\{c_1\}) \geq v_i(\{c_2\}) \geq \cdots \geq v_i(\{c_{|C_h|}\})$.*

Theorem 1. *For fair allocation under cardinality constraints, MMS-approximation reduces to MMS-approximation for ordered instances.*

4 REDUCED INSTANCES

If we remove an agent i and a bundle $B \subseteq M$ from an instance, the result is called a *reduced* instance. If the bundle's value is sufficiently high ($v_i(B) \geq \alpha\mu_i$) and the MMS of each remaining agent is at least as high after the removal, this is called a *valid reduction* [11], a concept used in many MMS approximation algorithms for the unconstrained fair allocation problem [e.g., 10, 12, 18].

Being able to guarantee that no agent i values any item at $\alpha\mu_i$ or higher limits the impact a single item can have on a bundle's value under additive valuations—a property that is important in our algorithm. For unconstrained instances, this property can easily be obtained, since the removal of an agent i and an item j with $v_i(\{j\}) \geq \alpha\mu_i$ is a valid reduction. Under cardinality constraints, however, a reduction with a single item may result in an instance with no feasible allocations. We can still construct a valid reduction, given a single item j worth at least $\alpha\mu_i$ to some agent i, as any bundle must contain at least $|C_h| - (n-1)k_h$ items from each category C_h. Thus, a bundle of j and the $|C_h \setminus \{j\}| - (n-1)k_h$ least valuable items of each category C_h forms a valid reduction.

Theorem 2. *Let $I = \langle N, M, V, C\rangle$ be an ordered instance of the fair allocation problem under cardinality constraints with an item j such that $v_i(\{j\}) \geq \alpha\mu_i$ for an agent i and an $\alpha > 0$. Let B be the bundle consisting of j and the $\max(0, |C_h \setminus \{j\}| - (n-1)k_h)$ least valuable items in each category C_h. Allocating B to i is a valid reduction for α.*

5 APPROXIMATION ALGORITHM

Theorems 1 and 2 simplify MMS approximation to the case of normalized ordered instances where no item is worth more than α. We now present an algorithm that for such an instance finds a 1/2-approximate MMS allocation (Algorithm 1). In the algorithm, let C_h^H and C_h^L denote the $\lfloor |C_h|/n \rfloor$ most and least valuable items, respectively, remaining in C_h.

Algorithm 1 Find a 1/2-MMS solution to an ordered instance

Input: A normalized ordered instance $I = \langle N, M, V, C\rangle$ with $v_i(\{j\}) < 1/2$ for all $i \in N$, $j \in M$
Output: Allocation A consisting of each bag B allocated

while there is more than one agent left
  $B = \bigcup_{h=1}^{\ell} C_h^L$
  while $v_i(B) < 1/2$ for all remaining agents i
    if $B \cap C_h^L \neq \emptyset$ for some C_h
      j = any element of $C_h^H \setminus B$
      j' = any element of $B \cap C_h^L$
      $B = (B \setminus \{j'\}) \cup \{j\}$
    else j = any $c_{\lceil |C_h|/n \rceil}$ not in B
      $B = B \cup \{j\}$
  allocate B to some agent i with $v_i(B) \geq 1/2$
  remove B and i from I and update C_h^H and C_h^L
allocate the remaining items to the last agent

The algorithm works in a similar manner to that of the bag filling algorithms in the unconstrained setting [see, e.g, 10, 12], i.e., by incrementally adding items to (and, in our case, removing items from) a "bag," B, until $v_i(B) \geq 1/2$ for some agent i. The major difference is the initial contents of the bag. As was the case in Section 4, we must make sure to not end up in a situation where a category C_h contains more items than can be given to the remaining agents. Thus, we start with a bag containing the $\lfloor |C_h|/n \rfloor$ least valuable items in each C_h. This guarantees that the bag contains enough items from each C_h, but is not worth more than $1/n$ of the remaining value. We then incrementally, so as not to increase the value by $1/2$ or more in a single step, exchange each of these lower-valued items for higher-valued items in the same C_h. If the bundle is still not worth $1/2$ to any agent, individual items are added, incrementally, from any C_h where the number of items is not exactly divisible by the number of remaining agents, i.e., where we earlier had to round down when selecting $1/n$ of the items. When this is done for all C_h, there must be an agent that values the bundle at least $1/2$. One can show that the value of each allocated bundle is no more than $1/n$ of the remaining value for all remaining agents. As a result, each agent receives a bundle, and it is worth at least half of its MMS.

Theorem 3. *Given a normalized ordered instance of the fair allocation problem under cardinality constraints where all items are worth less than $1/2$, Algorithm 1 finds a feasible $1/2$-approximate MMS allocation in polynomial time in the number of agents and items.*

6 DISCUSSION

Our algorithm finds 1/2-approximate MMS allocations under cardinality constraints. While an improvement to the previously best approximation guarantee, it is not unlikely that a better approximation guarantee is possible. However, different techniques to those in the unconstrained setting appear to be needed. It also remains unknown whether cardinality constraints have a stricter upper bound for MMS-approximation than in the unconstrained setting.

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