Actuarial Mathematics

Yosuke Ito

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Abstract

Actuarial Mathematics is a theory in applied mathematics, which is mainly used for determining the prices of insurance products and evaluating the liability of a company associating with insurance contracts. It is related to calculus, probability theory and financial theory, etc.

In this entry, I formalize the very basic part of Actuarial Mathematics in Isabelle/HOL. It includes the theory of interest, survival model, and life table. The theory of interest deals with interest rates, present value factors, an annuity certain, etc. The survival model is a probabilistic model that represents the human mortality. The life table is based on the survival model and used for practical calculations.

I have already formalized the basic part of Actuarial Mathematics in Coq (https://github.com/Yosuke-Ito-345/Actuary) in a purely axiomatic manner. In contrast, Isabelle formalization is based on the probability theory and the survival model is developed as generally as possible. Such rigorous and general formulation seems very rare; at least I cannot find any similar documentation on the web.

This formalization in Isabelle is still at an early stage, and I cannot guarantee the backward compatibility in the future development. If you heavily depend on the "Actuarial Mathematics" library, please let me know.

Contents

theory *Preliminaries*
imports *HOL−Library.Rewrite HOL−Library.Extended-Nonnegative-Real HOL−Library.Extended-Real HOL−Probability.Probability*
begin
declare [[*show-types*]]
notation *powr* (**infixr** .^ *80*)

1 Preliminary Lemmas

lemma *Collect-conj-eq2*: $\{x \in A.\ P\ x \wedge Q\ x\} = \{x \in A.\ P\ x\} \cap \{x \in A.\ Q\ x\}$
⟨*proof*⟩

lemma *vimage-compl-atMost*:
fixes $f :: 'a \Rightarrow 'b::linorder$
shows $-(f -` \{..y\}) = f -` \{y<..\}$
⟨*proof*⟩

context *linorder*
begin

lemma *Icc-minus-Ico*:
fixes $a\ b$
assumes $a \le b$
shows $\{a..b\} - \{a..<b\} = \{b\}$
⟨*proof*⟩

lemma *Icc-minus-Ioc*:
fixes $a\ b$
assumes $a \le b$
shows $\{a..b\} - \{a<..b\} = \{a\}$
⟨*proof*⟩

lemma *Int-atLeastAtMost-Unbounded*[*simp*]: $\{a..\}$ *Int* $\{..b\} = \{a..b\}$
⟨*proof*⟩

lemma *Int-greaterThanAtMost-Unbounded*[*simp*]: $\{a<..\}$ *Int* $\{..b\} = \{a<..b\}$
⟨*proof*⟩

lemma *Int-atLeastLessThan-Unbounded*[*simp*]: $\{a..\}$ *Int* $\{..<b\} = \{a..<b\}$
⟨*proof*⟩

lemma *Int-greaterThanLessThan-Unbounded*[*simp*]: $\{a<..\}$ *Int* $\{..<b\} = \{a<..<b\}$
⟨*proof*⟩

end

lemma *Ico-real-nat-disjoint*:
disjoint-family (λn::*nat*. $\{a + real\ n\ ..<\ a + real\ n + 1\}$) **for** *a*::*real*
⟨*proof*⟩

corollary *Ico-nat-disjoint*: *disjoint-family* (λn::*nat*. $\{real\ n\ ..<\ real\ n + 1\}$)
⟨*proof*⟩

lemma *Ico-real-nat-union*: ($\bigcup n$::*nat*. $\{a + real\ n\ ..<\ a + real\ n + 1\}$) = $\{a..\}$ **for** *a*::*real*
⟨*proof*⟩

corollary *Ico-nat-union*: ($\bigcup n$::*nat*. $\{real\ n\ ..<\ real\ n + 1\}$) = $\{0..\}$
⟨*proof*⟩

lemma *Ico-nat-union-finite*: ($\bigcup (n$::*nat*$)<m$. $\{real\ n\ ..<\ real\ n + 1\}$) = $\{0..<m\}$
⟨*proof*⟩

lemma *seq-part-multiple*: **fixes** $m\ n$ :: *nat* **assumes** $m \neq 0$ **defines** $A \equiv \lambda i$::*nat*. $\{i*m\ ..<\ (i+1)*m\}$
shows $\forall\, i\ j.\ i \neq j \longrightarrow A\ i \cap A\ j = \{\}$ **and** $(\bigcup i<n.\ A\ i) = \{..<\ n*m\}$
⟨*proof*⟩

lemma *frontier-Icc-real*: *frontier* $\{a..b\} = \{a,\ b\}$ **if** $a \leq b$ **for** $a\ b$ :: *real*
⟨*proof*⟩

lemma(**in** *field*) *divide-mult-cancel*[*simp*]: **fixes** $a\ b$ **assumes** $b \neq 0$
shows $a\ /\ b * b = a$
⟨*proof*⟩

lemma *inverse-powr*: $(1/a)$.^b = a.^$-b$ **if** $a > 0$ **for** $a\ b$ :: *real*
⟨*proof*⟩

lemma *powr-eq-one-iff-gen*[*simp*]: a.^$x = 1 \longleftrightarrow x = 0$ **if** $a > 0$ $a \neq 1$ **for** $a\ x$ :: *real*
⟨*proof*⟩

lemma *powr-less-cancel2*: $0 < a \Longrightarrow 0 < x \Longrightarrow 0 < y \Longrightarrow x$.^$a < y$.^$a \Longrightarrow x <$

y
for $a\ x\ y$ ::*real*
⟨*proof*⟩

lemma *geometric-increasing-sum-aux*: $(1-r)\hat{\ }2 * (\sum k<n.\ (k+1)*r\hat{\ }k) = 1 - (n+1)*r\hat{\ }n + n*r\hat{\ }(n+1)$
for n::*nat* **and** r::*real*
⟨*proof*⟩

lemma *geometric-increasing-sum*: $(\sum k<n.\ (k+1)*r\hat{\ }k) = (1 - (n+1)*r\hat{\ }n + n*r\hat{\ }(n+1)) / (1-r)\hat{\ }2$
if $r \neq 1$ **for** n::*nat* **and** r::*real*
⟨*proof*⟩

lemma *Reals-UNIV*[*simp*]: $\mathbb{R} = \{x$::*real*. *True*$\}$
⟨*proof*⟩

lemma *Lim-cong*:
assumes $\forall_F\ x\ in\ F.\ f\ x = g\ x$
shows $Lim\ F\ f = Lim\ F\ g$
⟨*proof*⟩

lemma *LIM-zero-iff'*: $((\lambda x.\ l - f\ x) \longrightarrow 0)\ F = (f \longrightarrow l)\ F$
for f :: $'a \Rightarrow\ 'b$::*real-normed-vector*
⟨*proof*⟩

lemma *antimono-onI*:
$(\bigwedge r\ s.\ r \in A \Longrightarrow s \in A \Longrightarrow r \leq s \Longrightarrow f\ r \geq f\ s) \Longrightarrow$ *antimono-on* $A\ f$
⟨*proof*⟩

lemma *antimono-onD*:
$[\![$*antimono-on* $A\ f;\ r \in A;\ s \in A;\ r \leq s]\!] \Longrightarrow f\ r \geq f\ s$
⟨*proof*⟩

lemma *antimono-imp-mono-on*: *antimono* $f \Longrightarrow$ *antimono-on* $A\ f$
⟨*proof*⟩

lemma *antimono-on-subset*: *antimono-on* $A\ f \Longrightarrow B \subseteq A \Longrightarrow$ *antimono-on* $B\ f$
⟨*proof*⟩

lemma *mono-on-antimono-on*:
fixes f :: $'a$::*order* $\Rightarrow\ 'b$::*ordered-ab-group-add*
shows *mono-on* $A\ f \longleftrightarrow$ *antimono-on* $A\ (\lambda r.\ -\ f\ r)$
⟨*proof*⟩

corollary *mono-antimono*:
fixes f :: $'a$::*order* $\Rightarrow\ 'b$::*ordered-ab-group-add*
shows *mono* $f \longleftrightarrow$ *antimono* $(\lambda r.\ -\ f\ r)$
⟨*proof*⟩

lemma *mono-on-at-top-le*:
fixes a :: $'a$::*linorder* **and** b :: $'b$::{*order-topology*, *linordered-ab-group-add*}
and f :: $'a \Rightarrow 'b$
assumes *f-mono*: *mono-on* $\{a..\}$ f **and** *f-to-l*: $(f \longrightarrow l)$ *at-top*
shows $\bigwedge x.\ x \in \{a..\} \Longrightarrow f\ x \leq l$
⟨*proof*⟩

corollary *mono-at-top-le*:
fixes b :: $'b$::{*order-topology*, *linordered-ab-group-add*} **and** f :: $'a$::*linorder* $\Rightarrow 'b$
assumes *mono* f **and** $(f \longrightarrow b)$ *at-top*
shows $\bigwedge x.\ f\ x \leq b$
⟨*proof*⟩

lemma *mono-on-at-bot-ge*:
fixes a :: $'a$::*linorder* **and** b :: $'b$::{*order-topology*, *linordered-ab-group-add*}
and f :: $'a \Rightarrow 'b$
assumes *f-mono*: *mono-on* $\{..a\}$ f **and** *f-to-l*: $(f \longrightarrow l)$ *at-bot*
shows $\bigwedge x.\ x \in \{..a\} \Longrightarrow f\ x \geq l$
⟨*proof*⟩

corollary *mono-at-bot-ge*:
fixes b :: $'b$::{*order-topology*, *linordered-ab-group-add*} **and** f :: $'a$::*linorder* $\Rightarrow 'b$
assumes *mono* f **and** $(f \longrightarrow b)$ *at-bot*
shows $\bigwedge x.\ f\ x \geq b$
⟨*proof*⟩

lemma *antimono-on-at-top-ge*:
fixes a :: $'a$::*linorder* **and** b :: $'b$::{*order-topology*, *linordered-ab-group-add*}
and f :: $'a \Rightarrow 'b$
assumes *f-antimono*: *antimono-on* $\{a..\}$ f **and** *f-to-l*: $(f \longrightarrow l)$ *at-top*
shows $\bigwedge x.\ x \in \{a..\} \Longrightarrow f\ x \geq l$
⟨*proof*⟩

corollary *antimono-at-top-le*:
fixes b :: $'b$::{*order-topology*, *linordered-ab-group-add*} **and** f :: $'a$::*linorder* $\Rightarrow 'b$
assumes *antimono* f **and** $(f \longrightarrow b)$ *at-top*
shows $\bigwedge x.\ f\ x \geq b$
⟨*proof*⟩

lemma *antimono-on-at-bot-ge*:
fixes a :: $'a$::*linorder* **and** b :: $'b$::{*order-topology*, *linordered-ab-group-add*}
and f :: $'a \Rightarrow 'b$
assumes *f-antimono*: *antimono-on* $\{..a\}$ f **and** *f-to-l*: $(f \longrightarrow l)$ *at-bot*
shows $\bigwedge x.\ x \in \{..a\} \Longrightarrow f\ x \leq l$
⟨*proof*⟩

corollary *antimono-at-bot-ge*:
fixes b :: $'b$::{*order-topology*, *linordered-ab-group-add*} **and** f :: $'a$::*linorder* $\Rightarrow 'b$

assumes *antimono f* **and** $(f \longrightarrow b)$ *at-bot*
shows $\bigwedge x.\ f\ x \leq b$
⟨*proof*⟩

lemma *continuous-cdivide*:
fixes *c*::*'a*::*real-normed-field*
assumes $c \neq 0$ *continuous F f*
shows *continuous F* $(\lambda x.\ f\ x\ /\ c)$
⟨*proof*⟩

lemma *continuous-mult-left-iff*:
fixes *c*::*'a*::*real-normed-field*
assumes $c \neq 0$
shows *continuous F f* $\longleftrightarrow$ *continuous F* $(\lambda x.\ c * f\ x)$
⟨*proof*⟩

lemma *continuous-mult-right-iff*:
fixes *c*::*'a*::*real-normed-field*
assumes $c \neq 0$
shows *continuous F f* $\longleftrightarrow$ *continuous F* $(\lambda x.\ f\ x * c)$
⟨*proof*⟩

lemma *continuous-cdivide-iff*:
fixes *c*::*'a*::*real-normed-field*
assumes $c \neq 0$
shows *continuous F f* $\longleftrightarrow$ *continuous F* $(\lambda x.\ f\ x\ /\ c)$
⟨*proof*⟩

lemma *continuous-cong*:
assumes *eventually* $(\lambda x.\ f\ x = g\ x)$ *F f* (*Lim F* $(\lambda x.\ x)$) $=$ *g* (*Lim F* $(\lambda x.\ x)$)
shows *continuous F f* $\longleftrightarrow$ *continuous F g*
⟨*proof*⟩

lemma *continuous-at-within-cong*:
assumes $f\ x = g\ x$ *eventually* $(\lambda x.\ f\ x = g\ x)$ (*at x within s*)
shows *continuous* (*at x within s*) *f* $\longleftrightarrow$ *continuous* (*at x within s*) *g*
⟨*proof*⟩

lemma *continuous-within-shift*:
fixes *a x* :: *'a* :: {*topological-ab-group-add*, *t2-space*}
and *s* :: *'a set*
and *f* :: *'a* ⇒ *'b*::*topological-space*
shows *continuous* (*at x within s*) $(\lambda x.\ f\ (x+a))$ $\longleftrightarrow$ *continuous* (*at* $(x+a)$ *within plus a ‘ s*) *f*
⟨*proof*⟩

lemma *isCont-shift*:
fixes *a x* :: *'a* :: {*topological-ab-group-add*, *t2-space*}
and *f* :: *'a* ⇒ *'b*::*topological-space*

shows $isCont\ (\lambda x.\ f\ (x{+}a))\ x \longleftrightarrow isCont\ f\ (x{+}a)$
⟨*proof*⟩

lemma *has-real-derivative-at-split*:
$(f\ has\text{-}real\text{-}derivative\ D)\ (at\ x) \longleftrightarrow$
$(f\ has\text{-}real\text{-}derivative\ D)\ (at\text{-}left\ x) \wedge (f\ has\text{-}real\text{-}derivative\ D)\ (at\text{-}right\ x)$
⟨*proof*⟩

lemma *DERIV-cmult-iff*:
assumes $c \neq 0$
shows $(f\ has\text{-}field\text{-}derivative\ D)\ (at\ x\ within\ s) \longleftrightarrow$
$((\lambda x.\ c * f\ x)\ has\text{-}field\text{-}derivative\ c * D)\ (at\ x\ within\ s)$
⟨*proof*⟩

lemma *DERIV-cmult-right-iff*:
assumes $c \neq 0$
shows $(f\ has\text{-}field\text{-}derivative\ D)\ (at\ x\ within\ s) \longleftrightarrow$
$((\lambda x.\ f\ x * c)\ has\text{-}field\text{-}derivative\ D * c)\ (at\ x\ within\ s)$
⟨*proof*⟩

lemma *DERIV-cdivide-iff*:
assumes $c \neq 0$
shows $(f\ has\text{-}field\text{-}derivative\ D)\ (at\ x\ within\ s) \longleftrightarrow$
$((\lambda x.\ f\ x\ /\ c)\ has\text{-}field\text{-}derivative\ D\ /\ c)\ (at\ x\ within\ s)$
⟨*proof*⟩

lemma *DERIV-ln-divide-chain*:
fixes $f :: real \Rightarrow real$
assumes $f\ x > 0$ **and** $(f\ has\text{-}real\text{-}derivative\ D)\ (at\ x\ within\ s)$
shows $((\lambda x.\ ln\ (f\ x))\ has\text{-}real\text{-}derivative\ (D\ /\ f\ x))\ (at\ x\ within\ s)$
⟨*proof*⟩

lemma *inverse-fun-has-integral-ln*:
fixes $f :: real \Rightarrow real$ **and** $f' :: real \Rightarrow real$
assumes $a \leq b$ **and**
$\bigwedge x.\ x \in \{a..b\} \Longrightarrow f\ x > 0$ **and**
$continuous\text{-}on\ \{a..b\}\ f$ **and**
$\bigwedge x.\ x \in \{a{<}..{<}b\} \Longrightarrow (f\ has\text{-}real\text{-}derivative\ f'\ x)\ (at\ x)$
shows $((\lambda x.\ f'\ x\ /\ f\ x)\ has\text{-}integral\ (ln\ (f\ b) - ln\ (f\ a)))\ \{a..b\}$
⟨*proof*⟩

lemma *DERIV-fun-powr2*:
fixes $a{::}real$
assumes *a-pos*: $a > 0$
and f: $DERIV\ f\ x :> r$
shows $DERIV\ (\lambda x.\ a.\widehat{\ }(f\ x))\ x :> a.\widehat{\ }(f\ x) * r * ln\ a$
⟨*proof*⟩

lemma *has-real-derivative-powr2*:

assumes *a-pos*: $a > 0$
shows $((\lambda x.\ a.\widehat{\ }x)$ *has-real-derivative* $a.\widehat{\ }x * ln\ a)\ (at\ x)$
⟨*proof*⟩

lemma *field-differentiable-shift*:
$(f$ *field-differentiable* $(at\ (x + z))) = ((\lambda x.\ f\ (x + z))$ *field-differentiable* $(at\ x))$
⟨*proof*⟩

1.1 Lemmas on *indicator* for a Linearly Ordered Type

lemma *indicator-Icc-shift*:
fixes $a\ b\ t\ x :: 'a::linordered\text{-}ab\text{-}group\text{-}add$
shows $indicator\ \{a..b\}\ x = indicator\ \{t+a..t+b\}\ (t+x)$
⟨*proof*⟩

lemma *indicator-Icc-shift-inverse*:
fixes $a\ b\ t\ x :: 'a::linordered\text{-}ab\text{-}group\text{-}add$
shows $indicator\ \{a-t..b-t\}\ x = indicator\ \{a..b\}\ (t+x)$
⟨*proof*⟩

lemma *indicator-Ici-shift*:
fixes $a\ t\ x :: 'a::linordered\text{-}ab\text{-}group\text{-}add$
shows $indicator\ \{a..\}\ x = indicator\ \{t+a..\}\ (t+x)$
⟨*proof*⟩

lemma *indicator-Ici-shift-inverse*:
fixes $a\ t\ x :: 'a::linordered\text{-}ab\text{-}group\text{-}add$
shows $indicator\ \{a-t..\}\ x = indicator\ \{a..\}\ (t+x)$
⟨*proof*⟩

lemma *indicator-Iic-shift*:
fixes $b\ t\ x :: 'a::linordered\text{-}ab\text{-}group\text{-}add$
shows $indicator\ \{..b\}\ x = indicator\ \{..t+b\}\ (t+x)$
⟨*proof*⟩

lemma *indicator-Iic-shift-inverse*:
fixes $b\ t\ x :: 'a::linordered\text{-}ab\text{-}group\text{-}add$
shows $indicator\ \{..b-t\}\ x = indicator\ \{..b\}\ (t+x)$
⟨*proof*⟩

lemma *indicator-Icc-reverse*:
fixes $a\ b\ t\ x :: 'a::linordered\text{-}ab\text{-}group\text{-}add$
shows $indicator\ \{a..b\}\ x = indicator\ \{t-b..t-a\}\ (t-x)$
⟨*proof*⟩

lemma *indicator-Icc-reverse-inverse*:
fixes $a\ b\ t\ x :: 'a::linordered\text{-}ab\text{-}group\text{-}add$
shows $indicator\ \{t-b..t-a\}\ x = indicator\ \{a..b\}\ (t-x)$

⟨*proof*⟩

lemma *indicator-Ici-reverse*:
fixes $a\ t\ x$ :: $'a$::*linordered-ab-group-add*
shows *indicator* $\{a..\}$ x = *indicator* $\{..t-a\}$ $(t-x)$
⟨*proof*⟩

lemma *indicator-Ici-reverse-inverse*:
fixes $b\ t\ x$ :: $'a$::*linordered-ab-group-add*
shows *indicator* $\{t-b..\}$ x = *indicator* $\{..b\}$ $(t-x)$
⟨*proof*⟩

lemma *indicator-Iic-reverse*:
fixes $b\ t\ x$ :: $'a$::*linordered-ab-group-add*
shows *indicator* $\{..b\}$ x = *indicator* $\{t-b..\}$ $(t-x)$
⟨*proof*⟩

lemma *indicator-Iic-reverse-inverse*:
fixes $a\ t\ x$ :: $'a$::*linordered-field*
shows *indicator* $\{..t-a\}$ x = *indicator* $\{a..\}$ $(t-x)$
⟨*proof*⟩

lemma *indicator-Icc-affine-pos*:
fixes $a\ b\ c\ t\ x$ :: $'a$::*linordered-field*
assumes $c > 0$
shows *indicator* $\{a..b\}$ x = *indicator* $\{t+c*a..t+c*b\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Icc-affine-pos-inverse*:
fixes $a\ b\ c\ t\ x$ :: $'a$::*linordered-field*
assumes $c > 0$
shows *indicator* $\{(a-t)/c..(b-t)/c\}$ x = *indicator* $\{a..b\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Ici-affine-pos*:
fixes $a\ c\ t\ x$ :: $'a$::*linordered-field*
assumes $c > 0$
shows *indicator* $\{a..\}$ x = *indicator* $\{t+c*a..\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Ici-affine-pos-inverse*:
fixes $a\ c\ t\ x$ :: $'a$::*linordered-field*
assumes $c > 0$
shows *indicator* $\{(a-t)/c..\}$ x = *indicator* $\{a..\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Iic-affine-pos*:
fixes $b\ c\ t\ x$ :: $'a$::*linordered-field*
assumes $c > 0$

shows *indicator* $\{..b\}$ x = *indicator* $\{..t+c*b\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Iic-affine-pos-inverse*:
fixes b c t x :: $'a$::*linordered-field*
assumes $c > 0$
shows *indicator* $\{..(b-t)/c\}$ x = *indicator* $\{..b\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Icc-affine-neg*:
fixes a b c t x :: $'a$::*linordered-field*
assumes $c < 0$
shows *indicator* $\{a..b\}$ x = *indicator* $\{t+c*b..t+c*a\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Icc-affine-neg-inverse*:
fixes a b c t x :: $'a$::*linordered-field*
assumes $c < 0$
shows *indicator* $\{(b-t)/c..(a-t)/c\}$ x = *indicator* $\{a..b\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Ici-affine-neg*:
fixes a c t x :: $'a$::*linordered-field*
assumes $c < 0$
shows *indicator* $\{a..\}$ x = *indicator* $\{..t+c*a\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Ici-affine-neg-inverse*:
fixes b c t x :: $'a$::*linordered-field*
assumes $c < 0$
shows *indicator* $\{(b-t)/c..\}$ x = *indicator* $\{..b\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Iic-affine-neg*:
fixes b c t x :: $'a$::*linordered-field*
assumes $c < 0$
shows *indicator* $\{..b\}$ x = *indicator* $\{t+c*b..\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Iic-affine-neg-inverse*:
fixes a c t x :: $'a$::*linordered-field*
assumes $c < 0$
shows *indicator* $\{..(a-t)/c\}$ x = *indicator* $\{a..\}$ $(t + c*x)$
⟨*proof*⟩

2 Additional Lemmas for the *HOL−Analysis* Library

lemma *differentiable-eq-field-differentiable-real*:
fixes f :: *real* ⇒ *real*

shows f *differentiable* $F \longleftrightarrow f$ *field-differentiable* F
⟨*proof*⟩

lemma *differentiable-on-eq-field-differentiable-real*:
fixes f :: *real* ⇒ *real*
shows f *differentiable-on* $s \longleftrightarrow (\forall x \in s.\ f$ *field-differentiable* $($*at* x *within* $s))$
⟨*proof*⟩

lemma *differentiable-on-cong* :
assumes $\bigwedge x.\ x \in s \Longrightarrow f\ x = g\ x$ **and** f *differentiable-on* s
shows g *differentiable-on* s
⟨*proof*⟩

lemma *C1-differentiable-imp-deriv-continuous-on*:
f *C1-differentiable-on* $S \Longrightarrow$ *continuous-on* S (*deriv* f)
⟨*proof*⟩

lemma *deriv-shift*:
assumes f *field-differentiable at* $(x+a)$
shows *deriv* $f\ (x+a) =$ *deriv* $(\lambda s.\ f\ (x+s))\ a$
⟨*proof*⟩

lemma *piecewise-differentiable-on-cong*:
assumes f *piecewise-differentiable-on* i
and $\bigwedge x.\ x \in i \Longrightarrow f\ x = g\ x$
shows g *piecewise-differentiable-on* i
⟨*proof*⟩

lemma *differentiable-piecewise*:
assumes *continuous-on* $i\ f$
and f *differentiable-on* i
shows f *piecewise-differentiable-on* i
⟨*proof*⟩

lemma *piecewise-differentiable-scaleR*:
assumes f *piecewise-differentiable-on* S
shows $(\lambda x.\ a *_R f\ x)$ *piecewise-differentiable-on* S
⟨*proof*⟩

lemma *differentiable-on-piecewise-compose*:
assumes f *piecewise-differentiable-on* S
and g *differentiable-on* f ‘ S
shows $g \circ f$ *piecewise-differentiable-on* S
⟨*proof*⟩

lemma *MVT-order-free*:
fixes $r\ h$ :: *real*
defines $I \equiv \{r..r+h\} \cup \{r+h..r\}$
assumes *continuous-on* $I\ f$ **and** f *differentiable-on interior* I

obtains t **where** $t \in \{0<..<1\}$ **and** $f\ (r+h) - f\ r = h * deriv\ f\ (r + t*h)$
$\langle proof \rangle$

lemma *integral-combine2*:
fixes $f :: real \Rightarrow 'a::banach$
assumes $a \leq c$ $c \leq b$
and f *integrable-on* $\{a..c\}$ f *integrable-on* $\{c..b\}$
shows $integral\ \{a..c\}\ f + integral\ \{c..b\}\ f = integral\ \{a..b\}\ f$
$\langle proof \rangle$

lemma *has-integral-null-interval*: **fixes** $a\ b :: real$ **and** $f::real \Rightarrow real$ **assumes** a
$\geq b$
shows (f *has-integral* 0) $\{a..b\}$
$\langle proof \rangle$

lemma *has-integral-interval-reverse*: **fixes** $f :: real \Rightarrow real$ **and** $a\ b :: real$
assumes $a \leq b$
and *continuous-on* $\{a..b\}$ f
shows $((\lambda x.\ f\ (a+b-x))$ *has-integral* $(integral\ \{a..b\}\ f))\ \{a..b\}$
$\langle proof \rangle$

lemma *FTC-real-deriv-has-integral*:
fixes $F :: real \Rightarrow real$
assumes $a \leq b$
and F *piecewise-differentiable-on* $\{a<..<b\}$
and *continuous-on* $\{a..b\}$ F
shows ($deriv\ F$ *has-integral* $F\ b - F\ a$) $\{a..b\}$
$\langle proof \rangle$

lemma *integrable-spike-cong*:
assumes *negligible* S $\bigwedge x.\ x \in T - S \Longrightarrow g\ x = f\ x$
shows f *integrable-on* $T \longleftrightarrow g$ *integrable-on* T
$\langle proof \rangle$

lemma *has-integral-powr2-from-0*:
fixes $a\ c :: real$
assumes *a-pos*: $a > 0$ **and** *a-neq-1*: $a \neq 1$ **and** *c-nneg*: $c \geq 0$
shows $((\lambda x.\ a.\widehat{}x)$ *has-integral* $((a.\widehat{}c - 1) / (ln\ a)))\ \{0..c\}$
$\langle proof \rangle$

lemma *integrable-on-powr2-from-0*:
fixes $a\ c :: real$
assumes *a-pos*: $a > 0$ **and** *a-neq-1*: $a \neq 1$ **and** *c-nneg*: $c \geq 0$
shows $(\lambda x.\ a.\widehat{}x)$ *integrable-on* $\{0..c\}$
$\langle proof \rangle$

lemma *integrable-on-powr2-from-0-general*:
fixes $a\ c :: real$
assumes *a-pos*: $a > 0$ **and** *c-nneg*: $c \geq 0$

shows $(\lambda x.\ a.\hat{\ }x)$ *integrable-on* $\{0..c\}$
⟨*proof*⟩

lemma *has-bochner-integral-power*:
fixes $a\ b$ :: *real* **and** k :: *nat*
assumes $a \leq b$
shows *has-bochner-integral lborel* $(\lambda x.\ x\hat{\ }k * \textit{indicator}\ \{a..b\}\ x)$ $((b\hat{\ }(k+1) - a\hat{\ }(k+1))\ /\ (k+1))$
⟨*proof*⟩

corollary *integrable-power*: $(a{::}real) \leq b \Longrightarrow$ *integrable lborel* $(\lambda x.\ x\hat{\ }k * \textit{indicator}\ \{a..b\}\ x)$
⟨*proof*⟩

lemma *has-integral-set-integral-real*:
fixes $f{::}'a{::}euclidean\text{-}space \Rightarrow real$ **and** $A :: 'a\ set$
assumes f: *set-integrable lborel A f*
shows (f *has-integral* (*set-lebesgue-integral lborel A f*)) A
⟨*proof*⟩

lemma *set-borel-measurable-lborel*:
set-borel-measurable lborel A f $\longleftrightarrow$ *set-borel-measurable borel A f*
⟨*proof*⟩

lemma *restrict-space-whole*[*simp*]: *restrict-space M* (*space M*) = M
⟨*proof*⟩

lemma *deriv-measurable-real*:
fixes $f :: real \Rightarrow real$
assumes *f differentiable-on S open S f* $\in$ *borel-measurable borel*
shows *set-borel-measurable borel S* (*deriv f*)
⟨*proof*⟩

lemma *piecewise-differentiable-on-deriv-measurable-real*:
fixes $f :: real \Rightarrow real$
assumes *f piecewise-differentiable-on S open S f* $\in$ *borel-measurable borel*
shows *set-borel-measurable borel S* (*deriv f*)
⟨*proof*⟩

lemma *borel-measurable-antimono*:
fixes $f :: real \Rightarrow real$
shows *antimono f* $\Longrightarrow f \in$ *borel-measurable borel*
⟨*proof*⟩

lemma *set-borel-measurable-restrict-space-iff*:
fixes $f :: 'a \Rightarrow 'b{::}real\text{-}normed\text{-}vector$
assumes Ω[*measurable*, *simp*]: $\Omega \cap$ *space M* $\in$ *sets M*

shows $f \in$ *borel-measurable* (*restrict-space* M Ω) $\longleftrightarrow$ *set-borel-measurable* M Ω f
⟨*proof*⟩

lemma *set-integrable-restrict-space-iff*:
fixes f :: $'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes $A \in$ *sets* M
shows *set-integrable* M A f $\longleftrightarrow$ *integrable* (*restrict-space* M A) f
⟨*proof*⟩

lemma *set-lebesgue-integral-restrict-space*:
fixes f :: $'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes $A \in$ *sets* M
shows *set-lebesgue-integral* M A f = *integral*^{L} (*restrict-space* M A) f
⟨*proof*⟩

lemma *distr-borel-lborel*: *distr* M *borel* f = *distr* M *lborel* f
⟨*proof*⟩

lemma *AE-translation*:
assumes *AE* x *in lborel*. P x **shows** *AE* x *in lborel*. P $(a+x)$
⟨*proof*⟩

lemma *set-AE-translation*:
assumes *AE* $x \in S$ *in lborel*. P x **shows** *AE* $x \in$ *plus* $(-a)$ ‘ S *in lborel*. P $(a+x)$
⟨*proof*⟩

lemma *AE-scale-measure-iff*:
assumes $r > 0$
shows (*AE* x *in* (*scale-measure* r M). P x) $\longleftrightarrow$ (*AE* x *in* M. P x)
⟨*proof*⟩

lemma *nn-set-integral-cong2*:
assumes *AE* $x \in A$ *in* M. f x = g x
shows $(\int^+ x \in A.\ f\ x\ \partial M) = (\int^+ x \in A.\ g\ x\ \partial M)$
⟨*proof*⟩

lemma *set-lebesgue-integral-cong-AE2*:
assumes [*measurable*]: $A \in$ *sets* M *set-borel-measurable* M A f *set-borel-measurable* M A g
assumes *AE* $x \in A$ *in* M. f x = g x
shows (*LINT* x:A|M. f x) = (*LINT* x:A|M. g x)
⟨*proof*⟩

proposition *set-nn-integral-eq-set-integral*:
assumes *AE* $x \in A$ *in* M. $0 \le f$ x *set-integrable* M A f
shows $(\int^+ x \in A.\ f\ x\ \partial M) = (\int x \in A.\ f\ x\ \partial M)$
⟨*proof*⟩

proposition *nn-integral-disjoint-family-on-finite*:
assumes [*measurable*]: $f \in$ *borel-measurable M* $\bigwedge(n::nat).\ n \in S \Longrightarrow B\ n \in$ *sets M*
and *disjoint-family-on B S finite S*
shows $(\int^{+} x \in (\bigcup n{\in}S.\ B\ n).\ f\ x\ \partial M) = (\sum n{\in}S.\ (\int^{+} x \in B\ n.\ f\ x\ \partial M))$
⟨*proof*⟩

lemma *nn-integral-distr-set*:
assumes $T \in$ *measurable M M'* **and** $f \in$ *borel-measurable* (*distr M M' T*)
and $A \in$ *sets M'* **and** $\bigwedge x.\ x \in$ *space M* $\Longrightarrow T\ x \in A$
shows *integral*^{N} (*distr M M' T*) f = *set-nn-integral* (*distr M M' T*) *A f*
⟨*proof*⟩

lemma *measure-eqI-Ioc*:
fixes *M N* :: *real measure*
assumes *sets*: *sets M* = *sets borel sets N* = *borel*
assumes *fin*: $\bigwedge a\ b.\ a \le b \Longrightarrow$ *emeasure M* $\{a{<}..b\} < \infty$
assumes *eq*: $\bigwedge a\ b.\ a \le b \Longrightarrow$ *emeasure M* $\{a{<}..b\}$ = *emeasure N* $\{a{<}..b\}$
shows $M = N$
⟨*proof*⟩

lemma (**in** *finite-measure*) *distributed-measure*:
assumes *distributed M N X f*
and $\bigwedge x.\ x \in$ *space N* $\Longrightarrow f\ x \ge 0$
and $A \in$ *sets N*
shows *measure M* (X −‘ $A \cap$ *space M*) = ($\int x.$ *indicator A x* $*$ $f\ x\ \partial N$)
⟨*proof*⟩

lemma *set-integrable-const*[*simp*]:
$A \in$ *sets M* $\Longrightarrow$ *emeasure M A* $< \infty \Longrightarrow$ *set-integrable M A* (λ-. *c*)
⟨*proof*⟩

lemma *set-integral-const*[*simp*]:
$A \in$ *sets M* $\Longrightarrow$ *emeasure M A* $< \infty \Longrightarrow$ *set-lebesgue-integral M A* (λ-. *c*) = *measure M A* $*_R$ *c*
⟨*proof*⟩

lemma *set-integral-empty-0*[*simp*]: *set-lebesgue-integral M* {} *f* = *0*
⟨*proof*⟩

lemma *set-integral-nonneg*[*simp*]:
fixes *f* :: *'a* ⇒ *real* **and** *A* :: *'a set*
shows ($\bigwedge x.\ x \in A \Longrightarrow 0 \le f\ x$) $\Longrightarrow 0 \le$ *set-lebesgue-integral M A f*
⟨*proof*⟩

lemma
fixes *f* :: *'a* ⇒ *'b*::{*banach*, *second-countable-topology*} **and** *w* :: *'a* ⇒ *real*

assumes $A \in$ *sets M set-borel-measurable M A f*
$\bigwedge i.$ *set-borel-measurable M A (s i) set-integrable M A w*
assumes *lim*: *AE x∈A in M.* $(\lambda i.\ s\ i\ x) \longrightarrow f\ x$
assumes *bound*: $\bigwedge i$::*nat. AE x∈A in M. norm* $(s\ i\ x) \leq w\ x$
shows *set-integrable-dominated-convergence*: *set-integrable M A f*
and *set-integrable-dominated-convergence2*: $\bigwedge i.$ *set-integrable M A (s i)*
and *set-integral-dominated-convergence*:
$(\lambda i.$ *set-lebesgue-integral M A (s i))* $\longrightarrow$ *set-lebesgue-integral M A f*
⟨*proof*⟩

lemma *absolutely-integrable-on-iff-set-integrable*:
fixes *f* :: *'a::euclidean-space ⇒ real*
assumes $f \in$ *borel-measurable lborel*
and $S \in$ *sets lborel*
shows *set-integrable lborel S f* $\longleftrightarrow$ *f absolutely-integrable-on S*
⟨*proof*⟩

corollary *integrable-on-iff-set-integrable-nonneg*:
fixes *f* :: *'a::euclidean-space ⇒ real*
assumes $\bigwedge x.\ x \in S \Longrightarrow f\ x \geq 0$ $f \in$ *borel-measurable lborel*
and $S \in$ *sets lborel*
shows *set-integrable lborel S f* $\longleftrightarrow$ *f integrable-on S*
⟨*proof*⟩

lemma *integrable-on-iff-set-integrable-nonneg-AE*:
fixes *f* :: *'a::euclidean-space ⇒ real*
assumes *AE x∈S in lborel.* $f\ x \geq 0$ $f \in$ *borel-measurable lborel*
and $S \in$ *sets lborel*
shows *set-integrable lborel S f* $\longleftrightarrow$ *f integrable-on S*
⟨*proof*⟩

lemma *set-borel-integral-eq-integral-nonneg*:
fixes *f* :: *'a::euclidean-space ⇒ real*
assumes $\bigwedge x.\ x \in S \Longrightarrow f\ x \geq 0$ $f \in$ *borel-measurable borel* $S \in$ *sets borel*
shows (*LINT x* : *S* | *lborel. f x*) = *integral S f*
— Note that $0 = 0$ holds when the integral diverges.
⟨*proof*⟩

lemma *set-borel-integral-eq-integral-nonneg-AE*:
fixes *f* :: *'a::euclidean-space ⇒ real*
assumes *AE x∈S in lborel.* $f\ x \geq 0$ $f \in$ *borel-measurable borel* $S \in$ *sets borel*
shows (*LINT x* : *S* | *lborel. f x*) = *integral S f*
— Note that $0 = 0$ holds when the integral diverges.
⟨*proof*⟩

2.1 Set Lebesgue Integrability on Affine Transformation

lemma *set-integrable-Icc-affine-pos-iff*:
fixes *f* :: *real ⇒ 'a::{banach, second-countable-topology}* **and** *a b c t* :: *real*

assumes *c* > *0*
shows *set-integrable lborel* {(*a*−*t*)/*c*..(*b*−*t*)/*c*} (λ*x*. *f* (*t* + *c*∗*x*))
⟷ *set-integrable lborel* {*a*..*b*} *f*
⟨*proof*⟩

corollary *set-integrable-Icc-shift*:
fixes *f* :: *real* ⇒ ′*a*::{*banach*, *second-countable-topology*} **and** *a b t* :: *real*
shows *set-integrable lborel* {*a*−*t*..*b*−*t*} (λ*x*. *f* (*t*+*x*)) ⟷ *set-integrable lborel* {*a*..*b*} *f*
⟨*proof*⟩

lemma *set-integrable-Ici-affine-pos-iff*:
fixes *f* :: *real* ⇒ ′*a*::{*banach*, *second-countable-topology*} **and** *a c t* :: *real*
assumes *c* > *0*
shows *set-integrable lborel* {(*a*−*t*)/*c*..} (λ*x*. *f* (*t* + *c*∗*x*))
⟷ *set-integrable lborel* {*a*..} *f*
⟨*proof*⟩

corollary *set-integrable-Ici-shift*:
fixes *f* :: *real* ⇒ ′*a*::{*banach*, *second-countable-topology*} **and** *a t* :: *real*
shows *set-integrable lborel* {*a*−*t*..} (λ*x*. *f* (*t*+*x*)) ⟷ *set-integrable lborel* {*a*..} *f*
⟨*proof*⟩

lemma *set-integrable-Iic-affine-pos-iff*:
fixes *f* :: *real* ⇒ ′*a*::{*banach*, *second-countable-topology*} **and** *b c t* :: *real*
assumes *c* > *0*
shows *set-integrable lborel* {..(*b*−*t*)/*c*} (λ*x*. *f* (*t* + *c*∗*x*))
⟷ *set-integrable lborel* {..*b*} *f*
⟨*proof*⟩

corollary *set-integrable-Iic-shift*:
fixes *f* :: *real* ⇒ ′*a*::{*banach*, *second-countable-topology*} **and** *b t* :: *real*
shows *set-integrable lborel* {..*b*−*t*} (λ*x*. *f* (*t*+*x*)) ⟷ *set-integrable lborel* {..*b*} *f*
⟨*proof*⟩

lemma *set-integrable-Icc-affine-neg-iff*:
fixes *f* :: *real* ⇒ ′*a*::{*banach*, *second-countable-topology*} **and** *a b c t* :: *real*
assumes *c* < *0*
shows *set-integrable lborel* {(*b*−*t*)/*c*..(*a*−*t*)/*c*} (λ*x*. *f* (*t* + *c*∗*x*))
⟷ *set-integrable lborel* {*a*..*b*} *f*
⟨*proof*⟩

corollary *set-integrable-Icc-reverse*:
fixes *f* :: *real* ⇒ ′*a*::{*banach*, *second-countable-topology*} **and** *a b t* :: *real*
shows *set-integrable lborel* {*t*−*b*..*t*−*a*} (λ*x*. *f* (*t*−*x*)) ⟷ *set-integrable lborel* {*a*..*b*} *f*
⟨*proof*⟩

lemma *set-integrable-Ici-affine-neg-iff*:

fixes f :: *real* ⇒ *'a*::{*banach*, *second-countable-topology*} **and** b c t :: *real*
assumes $c < 0$
shows *set-integrable lborel* {$(b-t)/c$..} ($\lambda x.$ f (t + $c*x$))
⟷ *set-integrable lborel* {..b} f
⟨*proof*⟩

corollary *set-integrable-Ici-reverse*:
fixes f :: *real* ⇒ *'a*::{*banach*, *second-countable-topology*} **and** b t :: *real*
shows *set-integrable lborel* {$t-b$..} ($\lambda x.$ f ($t-x$)) ⟷ *set-integrable lborel* {..b} f
⟨*proof*⟩

lemma *set-integrable-Iic-affine-neg-iff*:
fixes f :: *real* ⇒ *'a*::{*banach*, *second-countable-topology*} **and** a c t :: *real*
assumes $c < 0$
shows *set-integrable lborel* {..$(a-t)/c$} ($\lambda x.$ f (t + $c*x$))
⟷ *set-integrable lborel* {a..} f
⟨*proof*⟩

corollary *set-integrable-Iic-reverse*:
fixes f :: *real* ⇒ *'a*::{*banach*, *second-countable-topology*} **and** a t :: *real*
shows *set-integrable lborel* {..$t-a$} ($\lambda x.$ f ($t-x$)) ⟷ *set-integrable lborel* {a..} f
⟨*proof*⟩

2.2 Set Lebesgue Integral on Affine Transformation

lemma *lborel-set-integral-Icc-affine-pos*:
fixes f :: *real* ⇒ *'a* :: {*banach*, *second-countable-topology*} **and** a b c :: *real*
assumes $c > 0$
shows ($\int x \in \{a..b\}$. f x ∂*lborel*) = c $*_R$ ($\int x \in \{(a-t)/c..(b-t)/c\}$. f (t + $c*x$) ∂*lborel*)
⟨*proof*⟩

corollary *lborel-set-integral-Icc-shift*:
fixes f :: *real* ⇒ *'a* :: {*banach*, *second-countable-topology*} **and** a b :: *real*
shows ($\int x \in \{a..b\}$. f x ∂*lborel*) = ($\int x \in \{a-t..b-t\}$. f ($t+x$) ∂*lborel*)
⟨*proof*⟩

lemma *lborel-set-integral-Ici-affine-pos*:
fixes f :: *real* ⇒ *'a* :: {*banach*, *second-countable-topology*} **and** a c :: *real*
assumes $c > 0$
shows ($\int x \in \{a..\}$. f x ∂*lborel*) = c $*_R$ ($\int x \in \{(a-t)/c..\}$. f (t + $c*x$) ∂*lborel*)
⟨*proof*⟩

corollary *lborel-set-integral-Ici-shift*:
fixes f :: *real* ⇒ *'a* :: {*banach*, *second-countable-topology*} **and** a::*real*
shows ($\int x \in \{a..\}$. f x ∂*lborel*) = ($\int x \in \{a-t..\}$. f ($t+x$) ∂*lborel*)
⟨*proof*⟩

lemma *lborel-set-integral-Iic-affine-pos*:

fixes f :: *real* ⇒ *'a* :: {*banach*, *second-countable-topology*} **and** b c :: *real*
assumes $c > 0$
shows ($\int x \in \{..b\}$. f x ∂*lborel*) = $c *_R$ ($\int x \in \{..(b-t)/c\}$. f ($t + c*x$) ∂*lborel*)
⟨*proof*⟩

corollary *lborel-set-integral-Iic-shift*:
fixes f :: *real* ⇒ *'a* :: {*banach*, *second-countable-topology*} **and** b::*real*
shows ($\int x \in \{..b\}$. f x ∂*lborel*) = ($\int x \in \{..b-t\}$. f ($t+x$) ∂*lborel*)
⟨*proof*⟩

lemma *lborel-set-integral-Icc-affine-neg*:
fixes f :: *real* ⇒ *'a* :: {*banach*, *second-countable-topology*} **and** a b c :: *real*
assumes $c < 0$
shows ($\int x \in \{a..b\}$. f x ∂*lborel*) = $-c *_R$ ($\int x \in \{(b-t)/c..(a-t)/c\}$. f ($t + c*x$) ∂*lborel*)
⟨*proof*⟩

corollary *lborel-set-integral-Icc-reverse*:
fixes f :: *real* ⇒ *'a* :: {*banach*, *second-countable-topology*} **and** a b :: *real*
shows ($\int x \in \{a..b\}$. f x ∂*lborel*) = ($\int x \in \{t-b..t-a\}$. f ($t-x$) ∂*lborel*)
⟨*proof*⟩

lemma *lborel-set-integral-Ici-affine-neg*:
fixes f :: *real* ⇒ *'a* :: {*banach*, *second-countable-topology*} **and** b c :: *real*
assumes $c < 0$
shows ($\int x \in \{..b\}$. f x ∂*lborel*) = $-c *_R$ ($\int x \in \{(b-t)/c..\}$. f ($t + c*x$) ∂*lborel*)
⟨*proof*⟩

corollary *lborel-set-integral-Ici-reverse*:
fixes f :: *real* ⇒ *'a* :: {*banach*, *second-countable-topology*} **and** b::*real*
shows ($\int x \in \{..b\}$. f x ∂*lborel*) = ($\int x \in \{t-b..\}$. f ($t-x$) ∂*lborel*)
⟨*proof*⟩

lemma *lborel-set-integral-Iic-affine-neg*:
fixes f :: *real* ⇒ *'a* :: {*banach*, *second-countable-topology*} **and** a c :: *real*
assumes $c < 0$
shows ($\int x \in \{a..\}$. f x ∂*lborel*) = $-c *_R$ ($\int x \in \{..(a-t)/c\}$. f ($t + c*x$) ∂*lborel*)
⟨*proof*⟩

corollary *lborel-set-integral-Iic-reverse*:
fixes f :: *real* ⇒ *'a* :: {*banach*, *second-countable-topology*} **and** a::*real*
shows ($\int x \in \{a..\}$. f x ∂*lborel*) = ($\int x \in \{..t-a\}$. f ($t-x$) ∂*lborel*)
⟨*proof*⟩

lemma *set-integrable-Ici-equiv-aux*:
fixes f :: *real* ⇒ *'a*::{*banach*, *second-countable-topology*} **and** a b :: *real*
assumes $\bigwedge c\ d$. *set-integrable lborel* $\{c..d\}$ f $a \le b$
shows *set-integrable lborel* $\{a..\}$ f ⟷ *set-integrable lborel* $\{b..\}$ f
⟨*proof*⟩

corollary *set-integrable-Ici-equiv*:
fixes $f :: real \Rightarrow 'a::\{banach, second\text{-}countable\text{-}topology\}$ **and** $a\ b :: real$
assumes $\bigwedge c\ d.\ set\text{-}integrable\ lborel\ \{c..d\}\ f$
shows $set\text{-}integrable\ lborel\ \{a..\}\ f \longleftrightarrow set\text{-}integrable\ lborel\ \{b..\}\ f$
⟨*proof*⟩

lemma *set-integrable-Iic-equiv*:
fixes $f :: real \Rightarrow real$ **and** $a\ b :: real$
assumes $\bigwedge c\ d.\ set\text{-}integrable\ lborel\ \{c..d\}\ f$
shows $set\text{-}integrable\ lborel\ \{..a\}\ f \longleftrightarrow set\text{-}integrable\ lborel\ \{..b\}\ f$ (**is** $?LHS \longleftrightarrow ?RHS$)
⟨*proof*⟩

2.3 Alternative Integral Test

lemma *nn-integral-suminf-Ico-real-nat*:
fixes $a::real$ **and** $f :: real \Rightarrow ennreal$
assumes $f \in borel\text{-}measurable\ lborel$
shows $(\int^+ x\in\{a..\}.\ f\ x\ \partial lborel) = (\sum k.\ \int^+ x\in\{a+k..<a+k+1\}.\ f\ x\ \partial lborel)$
⟨*proof*⟩

lemma *set-integrable-iff-bounded*:
fixes $f :: 'a \Rightarrow 'b::\{banach, second\text{-}countable\text{-}topology\}$
assumes $A \in sets\ M$
shows $set\text{-}integrable\ M\ A\ f \longleftrightarrow set\text{-}borel\text{-}measurable\ M\ A\ f \wedge (\int^+ x\in A.\ norm\ (f\ x)\ \partial M) < \infty$
⟨*proof*⟩

theorem *set-integrable-iff-summable*:
fixes $a::real$ **and** $f :: real \Rightarrow real$
assumes $antimono\text{-}on\ \{a..\}\ f\ \bigwedge x.\ a \le x \Longrightarrow f\ x \ge 0\ f \in borel\text{-}measurable\ lborel$
shows $set\text{-}integrable\ lborel\ \{a..\}\ f \longleftrightarrow summable\ (\lambda k.\ f\ (a+k))$
⟨*proof*⟩

2.4 Interchange of Differentiation and Lebesgue Integration

definition $measurable\text{-}extension :: 'a\ measure \Rightarrow 'b\ measure \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$ **where**
$measurable\text{-}extension\ M\ N\ f =$
$(SOME\ g.\ g \in M \rightarrow_M N \wedge (\exists S\in(null\text{-}sets\ M).\ \{x \in space\ M.\ f\ x \neq g\ x\} \subseteq S))$
— The term *measurable-extension* is proposed by Reynald Affeldt.
— This function is used to make an almost-everywhere-defined function measurable.

lemma
fixes $f\ g$
assumes $g \in M \rightarrow_M N\ S \in null\text{-}sets\ M\ \{x \in space\ M.\ f\ x \neq g\ x\} \subseteq S$

shows *measurable-extensionI*: *AE x in M. f x = measurable-extension M N f x* **and**
measurable-extensionI2: *AE x in M. g x = measurable-extension M N f x* **and**
measurable-extension-measurable: *measurable-extension M N f ∈ measurable M N*
⟨*proof*⟩

corollary *measurable-measurable-extension-AE*:
fixes *f*
assumes $f \in M \to_M N$
shows *AE x in M. f x = measurable-extension M N f x*
⟨*proof*⟩

definition *borel-measurable-extension* ::
'a measure ⇒ ('a ⇒ 'b::topological-space) ⇒ 'a ⇒ 'b **where**
borel-measurable-extension M f = measurable-extension M borel f

lemma
fixes *f g*
assumes *g ∈ borel-measurable M S ∈ null-sets M {x ∈ space M. f x ≠ g x} ⊆ S*
shows *borel-measurable-extensionI*: *AE x in M. f x = borel-measurable-extension M f x* **and**
borel-measurable-extensionI2: *AE x in M. g x = borel-measurable-extension M f x* **and**
borel-measurable-extension-measurable: *borel-measurable-extension M f ∈ borel-measurable M*
⟨*proof*⟩

corollary *borel-measurable-measurable-extension-AE*:
fixes *f*
assumes *f ∈ borel-measurable M*
shows *AE x in M. f x = borel-measurable-extension M f x*
⟨*proof*⟩

definition *set-borel-measurable-extension* ::
'a measure ⇒ 'a set ⇒ ('a ⇒ 'b::topological-space) ⇒ 'a ⇒ 'b
where *set-borel-measurable-extension M A f = borel-measurable-extension (restrict-space M A) f*

lemma
fixes *f g :: 'a ⇒ 'b::real-normed-vector* **and** *A*
assumes *A ∈ sets M set-borel-measurable M A g S ∈ null-sets M {x ∈ A. f x ≠ g x} ⊆ S*
shows *set-borel-measurable-extensionI*:
AE x∈A in M. f x = set-borel-measurable-extension M A f x **and**
set-borel-measurable-extensionI2:
AE x∈A in M. g x = set-borel-measurable-extension M A f x **and**
set-borel-measurable-extension-measurable:
set-borel-measurable M A (set-borel-measurable-extension M A f)

⟨*proof*⟩

corollary *set-borel-measurable-measurable-extension-AE*:
fixes *f*::′*a* ⇒ ′*b*::*real-normed-vector* **and** *A*
assumes *set-borel-measurable M A f A* ∈ *sets M*
shows *AE x*∈*A in M*. *f x* = *set-borel-measurable-extension M A f x*
⟨*proof*⟩

proposition *interchange-deriv-LINT-general*:
fixes *a b* :: *real* **and** *f* :: *real* ⇒ ′*a* ⇒ *real* **and** *g* :: ′*a* ⇒ *real*
assumes *f-integ*: ⋀*r*. *r*∈{*a*<..<*b*} ⟹ *integrable M* (*f r*) **and**
f-diff: *AE x in M*. (λ*r*. *f r x*) *differentiable-on* {*a*<..<*b*} **and**
Df-bound: *AE x in M*. ∀ *r*∈{*a*<..<*b*}. |*deriv* (λ*r*. *f r x*) *r*| ≤ *g x integrable M g*
shows ⋀*r*. *r*∈{*a*<..<*b*} ⟹ ((λ*r*. ∫ *x*. *f r x* ∂*M*) *has-real-derivative*
∫ *x*. *borel-measurable-extension M* (λ*x*. *deriv* (λ*r*. *f r x*) *r*) *x* ∂*M*) (*at r*)
⟨*proof*⟩

proposition *interchange-deriv-LINT*:
fixes *a b* :: *real* **and** *f* :: *real* ⇒ ′*a* ⇒ *real* **and** *g* :: ′*a* ⇒ *real*
assumes ⋀*r*. *r*∈{*a*<..<*b*} ⟹ *integrable M* (*f r*) **and**
AE x in M. (λ*r*. *f r x*) *differentiable-on* {*a*<..<*b*} **and**
⋀*r*. *r*∈{*a*<..<*b*} ⟹ (λ*x*. (*deriv* (λ*r*. *f r x*) *r*)) ∈ *borel-measurable M* **and**
AE x in M. ∀ *r*∈{*a*<..<*b*}. |*deriv* (λ*r*. *f r x*) *r*| ≤ *g x integrable M g*
shows ⋀*r*. *r*∈{*a*<..<*b*} ⟹ ((λ*r*. ∫ *x*. *f r x* ∂*M*) *has-real-derivative*
∫ *x*. *deriv* (λ*r*. *f r x*) *r* ∂*M*) (*at r*)
⟨*proof*⟩

proposition *interchange-deriv-LINT-set-general*:
fixes *a b* :: *real* **and** *f* :: *real* ⇒ ′*a* ⇒ *real* **and** *g* :: ′*a* ⇒ *real* **and** *A* :: ′*a set*
assumes *A-msr*: *A* ∈ *sets M* **and**
f-integ: ⋀*r*. *r*∈{*a*<..<*b*} ⟹ *set-integrable M A* (*f r*) **and**
f-diff: *AE x*∈*A in M*. (λ*r*. *f r x*) *differentiable-on* {*a*<..<*b*} **and**
Df-bound: *AE x*∈*A in M*. ∀ *r*∈{*a*<..<*b*}. |*deriv* (λ*r*. *f r x*) *r*| ≤ *g x set-integrable*
M A g
shows ⋀*r*. *r*∈{*a*<..<*b*} ⟹ ((λ*r*. ∫ *x*∈*A*. *f r x* ∂*M*) *has-real-derivative*
(∫ *x*∈*A*. *set-borel-measurable-extension M A* (λ*x*. *deriv* (λ*r*. *f r x*) *r*) *x* ∂*M*))
(*at r*)
⟨*proof*⟩

proposition *interchange-deriv-LINT-set*:
fixes *a b* :: *real* **and** *f* :: *real* ⇒ ′*a* ⇒ *real* **and** *g* :: ′*a* ⇒ *real* **and** *A* :: ′*a set*
assumes *A* ∈ *sets M* **and**
⋀*r*. *r*∈{*a*<..<*b*} ⟹ *set-integrable M A* (*f r*) **and**
AE x∈*A in M*. (λ*r*. *f r x*) *differentiable-on* {*a*<..<*b*} **and**
⋀*r*. *r*∈{*a*<..<*b*} ⟹ *set-borel-measurable M A* (λ*x*. (*deriv* (λ*r*. *f r x*) *r*)) **and**
AE x∈*A in M*. ∀ *r*∈{*a*<..<*b*}. |*deriv* (λ*r*. *f r x*) *r*| ≤ *g x set-integrable M A g*
shows ⋀*r*. *r*∈{*a*<..<*b*} ⟹ ((λ*r*. ∫ *x*∈*A*. *f r x* ∂*M*) *has-real-derivative*
(∫ *x*∈*A*. *deriv* (λ*r*. *f r x*) *r* ∂*M*)) (*at r*)
⟨*proof*⟩

3 Additional Lemmas for the *HOL−Probability* Library

lemma (**in** *finite-borel-measure*)
  fixes *F* :: *real* ⇒ *real*
  assumes *nondecF* : $\bigwedge x\ y.\ x \leq y \Longrightarrow F\ x \leq F\ y$ **and**
    right-cont-F : $\bigwedge a.$ *continuous* (*at-right a*) *F* **and**
    lim-F-at-bot : $(F \longrightarrow 0)$ *at-bot* **and**
    lim-F-at-top : $(F \longrightarrow m)$ *at-top* **and**
    m : $0 \leq m$
  shows *emeasure-interval-measure-Ioi*: *emeasure* (*interval-measure F*) $\{x<..\} = m - F\ x$
    and *measure-interval-measure-Ioi*: *measure* (*interval-measure F*) $\{x<..\} = m - F\ x$
⟨*proof*⟩

lemma (**in** *prob-space*) *cond-prob-nonneg*[*simp*]: *cond-prob M P Q* ≥ 0
  ⟨*proof*⟩

lemma (**in** *prob-space*) *cond-prob-whole-1*: *cond-prob M P P* $= 1$ **if** *prob* $\{\omega \in$ *space M*. $P\ \omega\} \neq 0$
  ⟨*proof*⟩

lemma (**in** *prob-space*) *cond-prob-0-null*: *cond-prob M P Q* $= 0$ **if** *prob* $\{\omega \in$ *space M*. $Q\ \omega\} = 0$
  ⟨*proof*⟩

lemma (**in** *prob-space*) *cond-prob-AE-prob*:
  assumes $\{\omega \in$ *space M*. $P\ \omega\} \in$ *events* $\{\omega \in$ *space M*. $Q\ \omega\} \in$ *events*
    and *AE* ω *in M*. $Q\ \omega$
  shows *cond-prob M P Q* = *prob* $\{\omega \in$ *space M*. $P\ \omega\}$
⟨*proof*⟩

3.1 More Properties of *cdf*'s

context *finite-borel-measure*
begin

lemma *cdf-diff-eq2*:
  assumes $x \leq y$
  shows *cdf M y* − *cdf M x* = *measure M* $\{x<..y\}$
⟨*proof*⟩

end

context *prob-space*
begin

lemma *cdf-distr-measurable* [*measurable*]:

assumes [*measurable*]: *random-variable borel X*
shows *cdf* (*distr M borel X*) $\in$ *borel-measurable borel*
⟨*proof*⟩

lemma *cdf-distr-P*:
assumes *random-variable borel X*
shows *cdf* (*distr M borel X*) $x = \mathcal{P}(\omega$ *in M*. $X\ \omega \leq x)$
⟨*proof*⟩

lemma *cdf-continuous-distr-P-lt*:
assumes *random-variable borel X isCont* (*cdf* (*distr M borel X*)) *x*
shows *cdf* (*distr M borel X*) $x = \mathcal{P}(\omega$ *in M*. $X\ \omega < x)$
⟨*proof*⟩

lemma *cdf-distr-diff-P*:
assumes $x \leq y$
and *random-variable borel X*
shows *cdf* (*distr M borel X*) $y -$ *cdf* (*distr M borel X*) $x = \mathcal{P}(\omega$ *in M*. $x < X\ \omega \wedge X\ \omega \leq y)$
⟨*proof*⟩

lemma *cdf-distr-max*:
fixes *c*::*real*
assumes [*measurable*]: *random-variable borel X*
shows *cdf* (*distr M borel* (λx. *max* (*X x*) *c*)) *u* = *cdf* (*distr M borel X*) *u* $*$ *indicator* {*c*..} *u*
⟨*proof*⟩

lemma *cdf-distr-min*:
fixes *c*::*real*
assumes [*measurable*]: *random-variable borel X*
shows *cdf* (*distr M borel* (λx. *min* (*X x*) *c*)) *u* =
1 − (*1* − *cdf* (*distr M borel X*) *u*) $*$ *indicator* {..<*c*} *u*
⟨*proof*⟩

lemma *cdf-distr-floor-P*:
fixes $X :: {}'a \Rightarrow real$
assumes [*measurable*]: *random-variable borel X*
shows *cdf* (*distr M borel* (λx. $\lfloor X\ x \rfloor$)) $u = \mathcal{P}(x$ *in M*. $X\ x < \lfloor u \rfloor + 1)$
⟨*proof*⟩

lemma *expectation-nonneg-tail*:
assumes [*measurable*]: *random-variable borel X*
and *X-nonneg*: $\bigwedge x.\ x \in$ *space M* $\Longrightarrow X\ x \geq 0$
defines $F\ u \equiv$ *cdf* (*distr M borel X*) *u*
shows ($\int^+ x$. *ennreal* (*X x*) ∂M) = ($\int^+ u \in \{0..\}$. *ennreal* (*1* − *F u*) ∂*lborel*)
⟨*proof*⟩

lemma *expectation-nonpos-tail*:

assumes [*measurable*]: *random-variable borel X*
and *X-nonpos*: $\bigwedge x.\ x \in space\ M \Longrightarrow X\ x \leq 0$
defines $F\ u \equiv cdf\ (distr\ M\ borel\ X)\ u$
shows $(\int^+ x.\ ennreal\ (-\ X\ x)\ \partial M) = (\int^+ u{\in}\{..0\}.\ ennreal\ (F\ u)\ \partial lborel)$
⟨*proof*⟩

proposition *expectation-tail*:
assumes [*measurable*]: *integrable M X*
defines $F\ u \equiv cdf\ (distr\ M\ borel\ X)\ u$
shows $expectation\ X = (LBINT\ u{:}\{0..\}.\ 1\ -\ F\ u) - (LBINT\ u{:}\{..0\}.\ F\ u)$
⟨*proof*⟩

proposition *distributed-deriv-cdf*:
assumes [*measurable*]: *random-variable borel X*
defines $F\ u \equiv cdf\ (distr\ M\ borel\ X)\ u$
assumes $finite\ S\ \bigwedge x.\ x \notin S \Longrightarrow (F\ has\text{-}real\text{-}derivative\ f\ x)\ (at\ x)$
and $continuous\text{-}on\ UNIV\ F\ f \in borel\text{-}measurable\ lborel$
shows *distributed M lborel X f*
⟨*proof*⟩

end

Define the conditional probability space. This is obtained by rescaling the restricted space of a probability space.

3.2 Conditional Probability Space

lemma *restrict-prob-space*:
assumes $measure\text{-}space\ \Omega\ A\ \mu\ a \in A$
and $0 < \mu\ a\ \mu\ a < \infty$
shows $prob\text{-}space\ (scale\text{-}measure\ (1\ /\ \mu\ a)\ (restrict\text{-}space\ (measure\text{-}of\ \Omega\ A\ \mu)\ a))$
⟨*proof*⟩

definition *cond-prob-space* :: *'a measure ⇒ 'a set ⇒ 'a measure* (**infix** ⌊ *200*)
where $M{\lfloor}A \equiv scale\text{-}measure\ (1\ /\ emeasure\ M\ A)\ (restrict\text{-}space\ M\ A)$

context *prob-space*
begin

lemma *cond-prob-space-whole*[*simp*]: $M\ \lfloor\ space\ M = M$
⟨*proof*⟩

lemma *cond-prob-space-correct*:
assumes $A \in events\ prob\ A > 0$
shows $prob\text{-}space\ (M{\lfloor}A)$
⟨*proof*⟩

lemma *space-cond-prob-space*:

assumes $A \in events$
shows $space\ (M\downharpoonright A) = A$
⟨*proof*⟩

lemma *sets-cond-prob-space*: $sets\ (M\downharpoonright A) = (\cap)\ A\ `\ events$
⟨*proof*⟩

lemma *measure-cond-prob-space*:
assumes $A \in events\ B \in events$
and $prob\ A > 0$
shows $measure\ (M\downharpoonright A)\ (B \cap A) = prob\ (B \cap A)\ /\ prob\ A$
⟨*proof*⟩

corollary *measure-cond-prob-space-subset*:
assumes $A \in events\ B \in events\ B \subseteq A$
and $prob\ A > 0$
shows $measure\ (M\downharpoonright A)\ B = prob\ B\ /\ prob\ A$
⟨*proof*⟩

lemma *cond-cond-prob-space*:
assumes $A \in events\ B \in events\ B \subseteq A\ prob\ B > 0$
shows $(M\downharpoonright A)\downharpoonright B = M\downharpoonright B$
⟨*proof*⟩

lemma *cond-prob-space-prob*:
assumes[*measurable*]: $Measurable.pred\ M\ P\ Measurable.pred\ M\ Q$
and $\mathcal{P}(x\ in\ M.\ Q\ x) > 0$
shows $measure\ (M \downharpoonright \{x \in space\ M.\ Q\ x\})\ \{x \in space\ M.\ P\ x \wedge Q\ x\} = \mathcal{P}(x\ in\ M.\ P\ x \mid Q\ x)$
⟨*proof*⟩

lemma *cond-prob-space-cond-prob*:
assumes [*measurable*]: $Measurable.pred\ M\ P\ Measurable.pred\ M\ Q$
and $\mathcal{P}(x\ in\ M.\ Q\ x) > 0$
shows $\mathcal{P}(x\ in\ M.\ P\ x \mid Q\ x) = \mathcal{P}(x\ in\ (M \downharpoonright \{x \in space\ M.\ Q\ x\}).\ P\ x)$
⟨*proof*⟩

lemma *cond-prob-neg*:
assumes[*measurable*]: $Measurable.pred\ M\ P\ Measurable.pred\ M\ Q$
and $\mathcal{P}(x\ in\ M.\ Q\ x) > 0$
shows $\mathcal{P}(x\ in\ M.\ \neg\ P\ x \mid Q\ x) = 1 - \mathcal{P}(x\ in\ M.\ P\ x \mid Q\ x)$
⟨*proof*⟩

lemma *random-variable-cond-prob-space*:
assumes $A \in events\ prob\ A > 0$
and [*measurable*]: $random\text{-}variable\ borel\ X$
shows $X \in borel\text{-}measurable\ (M\downharpoonright A)$
⟨*proof*⟩

lemma *AE-cond-prob-space-iff*:
  assumes $A \in$ *events prob* $A > 0$
  shows $(AE\ x\ in\ M{\downharpoonright}A.\ P\ x) \longleftrightarrow (AE\ x\ in\ M.\ x \in A \longrightarrow P\ x)$
⟨*proof*⟩

lemma *integral-cond-prob-space-nn*:
  assumes $A \in$ *events prob* $A > 0$
    and [*measurable*]: *random-variable borel X*
    and *nonneg*: $AE\ x\ in\ M.\ x \in A \longrightarrow 0 \leq X\ x$
  shows $integral^L\ (M{\downharpoonright}A)\ X = expectation\ (\lambda x.\ indicator\ A\ x * X\ x)\ /\ prob\ A$
⟨*proof*⟩

end

Define the complementary cumulative distribution function, also known as tail distribution. The theory presented below is a slight modification of the subsection "Properties of cdf's" in the theory *Distribution-Functions*.

3.3 Complementary Cumulative Distribution Function

definition *ccdf* :: *real measure* ⇒ *real* ⇒ *real*
  where $ccdf\ M \equiv \lambda x.\ measure\ M\ \{x{<}..\}$
    — complementary cumulative distribution function (tail distribution)

lemma *ccdf-def2*: $ccdf\ M\ x = measure\ M\ \{x{<}..\}$
  ⟨*proof*⟩

context *finite-borel-measure*
begin

lemma *add-cdf-ccdf*: $cdf\ M\ x + ccdf\ M\ x = measure\ M\ (space\ M)$
⟨*proof*⟩

lemma *ccdf-cdf*: $ccdf\ M = (\lambda x.\ measure\ M\ (space\ M) - cdf\ M\ x)$
  ⟨*proof*⟩

lemma *cdf-ccdf*: $cdf\ M = (\lambda x.\ measure\ M\ (space\ M) - ccdf\ M\ x)$
  ⟨*proof*⟩

lemma *isCont-cdf-ccdf*: $isCont\ (cdf\ M)\ x \longleftrightarrow isCont\ (ccdf\ M)\ x$
⟨*proof*⟩

lemma *isCont-ccdf*: $isCont\ (ccdf\ M)\ x \longleftrightarrow measure\ M\ \{x\} = 0$
  ⟨*proof*⟩

lemma *continuous-cdf-ccdf*:
  shows $continuous\ F\ (cdf\ M) \longleftrightarrow continuous\ F\ (ccdf\ M)$
    (**is** $?LHS \longleftrightarrow ?RHS$)
⟨*proof*⟩

lemma *has-real-derivative-cdf-ccdf*:
($cdf\ M$ *has-real-derivative* D) $F \longleftrightarrow$ ($ccdf\ M$ *has-real-derivative* $-D$) F
⟨*proof*⟩

lemma *differentiable-cdf-ccdf*: ($cdf\ M$ *differentiable* F) $\longleftrightarrow$ ($ccdf\ M$ *differentiable* F)
⟨*proof*⟩

lemma *deriv-cdf-ccdf*:
assumes *cdf M differentiable at x*
shows *deriv* ($cdf\ M$) $x = -$ *deriv* ($ccdf\ M$) x
⟨*proof*⟩

lemma *ccdf-diff-eq2*:
assumes $x \leq y$
shows $ccdf\ M\ x - ccdf\ M\ y = measure\ M\ \{x<..y\}$
⟨*proof*⟩

lemma *ccdf-nonincreasing*: $x \leq y \Longrightarrow ccdf\ M\ x \geq ccdf\ M\ y$
⟨*proof*⟩

lemma *ccdf-nonneg*: $ccdf\ M\ x \geq 0$
⟨*proof*⟩

lemma *ccdf-bounded*: $ccdf\ M\ x \leq measure\ M\ (space\ M)$
⟨*proof*⟩

lemma *ccdf-lim-at-top*: ($ccdf\ M \longrightarrow 0$) *at-top*
⟨*proof*⟩

lemma *ccdf-lim-at-bot*: ($ccdf\ M \longrightarrow measure\ M\ (space\ M)$) *at-bot*
⟨*proof*⟩

lemma *ccdf-is-right-cont*: *continuous* (*at-right a*) ($ccdf\ M$)
⟨*proof*⟩

end

context *prob-space*
begin

lemma *ccdf-distr-measurable* [*measurable*]:
assumes [*measurable*]: *random-variable borel X*
shows *ccdf* (*distr M borel X*) $\in$ *borel-measurable borel*
⟨*proof*⟩

lemma *ccdf-distr-P*:
assumes *random-variable borel X*

shows *ccdf* (*distr M borel X*) $x = \mathcal{P}(\omega\ in\ M.\ X\ \omega > x)$
⟨*proof*⟩

lemma *ccdf-continuous-distr-P-ge*:
assumes *random-variable borel X isCont* (*ccdf* (*distr M borel X*)) *x*
shows *ccdf* (*distr M borel X*) $x = \mathcal{P}(\omega\ in\ M.\ X\ \omega \geq x)$
⟨*proof*⟩

lemma *ccdf-distr-diff-P*:
assumes $x \leq y$
and *random-variable borel X*
shows *ccdf* (*distr M borel X*) $x -$ *ccdf* (*distr M borel X*) $y = \mathcal{P}(\omega\ in\ M.\ x < X\ \omega \wedge X\ \omega \leq y)$
⟨*proof*⟩

end

context *real-distribution*
begin

lemma *ccdf-bounded-prob*: $\bigwedge x.$ *ccdf M* $x \leq 1$
⟨*proof*⟩

lemma *ccdf-lim-at-bot-prob*: (*ccdf M* $\longrightarrow$ *1*) *at-bot*
⟨*proof*⟩

end

Introduce the hazard rate. This notion will be used to define the force of mortality.

3.4 Hazard Rate

context *prob-space*
begin

definition *hazard-rate* :: ($'a \Rightarrow$ *real*) $\Rightarrow$ *real* $\Rightarrow$ *real*
where *hazard-rate X t* $\equiv$
Lim (*at-right 0*) ($\lambda dt.\ \mathcal{P}(x\ in\ M.\ t < X\ x \wedge X\ x \leq t + dt \mid X\ x > t)\ /\ dt$)
— Here, *X* is supposed to be a random variable.

lemma *hazard-rate-0-ccdf-0*:
assumes [*measurable*]: *random-variable borel X*
and *ccdf* (*distr M borel X*) $t = 0$
shows *hazard-rate X t* $= 0$
— Note that division by *0* is calculated as *0* in Isabelle/HOL.
⟨*proof*⟩

lemma *hazard-rate-deriv-cdf*:

assumes [*measurable*]: *random-variable borel X*
and (*cdf* (*distr M borel X*)) *differentiable at t*
shows *hazard-rate X t* = *deriv* (*cdf* (*distr M borel X*)) *t* / *ccdf* (*distr M borel X*) *t*
⟨*proof*⟩

lemma *deriv-cdf-hazard-rate*:
assumes [*measurable*]: *random-variable borel X*
and (*cdf* (*distr M borel X*)) *differentiable at t*
shows *deriv* (*cdf* (*distr M borel X*)) *t* = *ccdf* (*distr M borel X*) *t* ∗ *hazard-rate X t*
⟨*proof*⟩

lemma *hazard-rate-deriv-ccdf*:
assumes [*measurable*]: *random-variable borel X*
and (*ccdf* (*distr M borel X*)) *differentiable at t*
shows *hazard-rate X t* = − *deriv* (*ccdf* (*distr M borel X*)) *t* / *ccdf* (*distr M borel X*) *t*
⟨*proof*⟩

lemma *deriv-ccdf-hazard-rate*:
assumes [*measurable*]: *random-variable borel X*
and (*ccdf* (*distr M borel X*)) *differentiable at t*
shows *deriv* (*ccdf* (*distr M borel X*)) *t* = − *ccdf* (*distr M borel X*) *t* ∗ *hazard-rate X t*
⟨*proof*⟩

lemma *hazard-rate-deriv-ln-ccdf*:
assumes [*measurable*]: *random-variable borel X*
and (*ccdf* (*distr M borel X*)) *differentiable at t*
and *ccdf* (*distr M borel X*) *t* ≠ *0*
shows *hazard-rate X t* = − *deriv* (λ*t*. *ln* (*ccdf* (*distr M borel X*) *t*)) *t*
⟨*proof*⟩

lemma *hazard-rate-has-integral*:
assumes [*measurable*]: *random-variable borel X*
and *t* ≤ *u*
and (*ccdf* (*distr M borel X*)) *piecewise-differentiable-on* {*t*<..<*u*}
and *continuous-on* {*t*..*u*} (*ccdf* (*distr M borel X*))
and ⋀*s*. *s* ∈ {*t*..*u*} ⟹ *ccdf* (*distr M borel X*) *s* ≠ *0*
shows
(*hazard-rate X has-integral ln* (*ccdf* (*distr M borel X*) *t* / *ccdf* (*distr M borel X*) *u*)) {*t*..*u*}
⟨*proof*⟩

corollary *hazard-rate-integrable*:
assumes [*measurable*]: *random-variable borel X*
and *t* ≤ *u*
and (*ccdf* (*distr M borel X*)) *piecewise-differentiable-on* {*t*<..<*u*}

and *continuous-on* $\{t..u\}$ (*ccdf* (*distr M borel X*))
and $\bigwedge s.\ s \in \{t..u\} \Longrightarrow$ *ccdf* (*distr M borel X*) $s \neq 0$
shows *hazard-rate X integrable-on* $\{t..u\}$
⟨*proof*⟩

lemma *ccdf-exp-cumulative-hazard*:
assumes [*measurable*]: *random-variable borel X*
and $t \leq u$
and (*ccdf* (*distr M borel X*)) *piecewise-differentiable-on* $\{t<..<u\}$
and *continuous-on* $\{t..u\}$ (*ccdf* (*distr M borel X*))
and $\bigwedge s.\ s \in \{t..u\} \Longrightarrow$ *ccdf* (*distr M borel X*) $s \neq 0$
shows *ccdf* (*distr M borel X*) u / *ccdf* (*distr M borel X*) t =
exp (− *integral* $\{t..u\}$ (*hazard-rate X*))
⟨*proof*⟩

lemma *hazard-rate-density-ccdf*:
assumes *distributed M lborel X f*
and $\bigwedge s.\ f\ s \geq 0$ $t < u$ *continuous-on* $\{t..u\}$ f
shows *hazard-rate X* $t = f\ t$ / *ccdf* (*distr M borel X*) t
⟨*proof*⟩

end

end
theory *Interest*
imports *Preliminaries*
begin

4 Theory of Interest

locale *interest* =
fixes i :: *real* — i stands for an interest rate.
assumes *v-futr-pos*: $1 + i > 0$ — Assume that the future value is positive.
begin

definition *i-nom* :: *nat* ⇒ *real* ($\$i\hat{}$\{-\} [0] 200)
where $\$i\hat{}\{m\} \equiv m * ((1+i).\hat{}(1/m) - 1)$ — nominal interest rate

definition *i-force* :: *real* ($\$\delta$ 200)
where $\$\delta \equiv ln\ (1+i)$ — force of interest

definition *d-nom* :: *nat* ⇒ *real* ($\$d\hat{}$\{-\} [0] 200)
where $\$d\hat{}\{m\} \equiv \$i\hat{}\{m\}\ /\ (1 + \$i\hat{}\{m\}/m)$ — discount rate

abbreviation *d-nom-yr* :: *real* ($\$d$ 200)
where $\$d \equiv \$d\hat{}\{1\}$ — Post-fix *yr* stands for "year".

definition *v-pres* :: *real* ($\$v$ 200)
where $\$v \equiv 1\ /\ (1+i)$ — present value factor

definition *ann* :: *nat* ⇒ *nat* ⇒ *real* ($*a*^{-}′-- [*0*,*101*] *200*)
where $*a*^{*m*}-*n* ≡ ∑ *k*<*n*∗*m*. $*v*.^((*k*+*1*::*nat*)/*m*) / *m*
— present value of an immediate annuity

abbreviation *ann-yr* :: *nat* ⇒ *real* ($*a*′-- [*101*] *200*)
where $*a*-*n* ≡ $*a*^{*1*}-*n*

definition *acc* :: *nat* ⇒ *nat* ⇒ *real* ($*s*^{-}′-- [*0*,*101*] *200*)
where $*s*^{*m*}-*n* ≡ ∑ *k*<*n*∗*m*. (*1*+*i*).^((*k*::*nat*)/*m*) / *m*
— future value of an immediate annuity
— The name *acc* stands for "accumulation".

abbreviation *acc-yr* :: *nat* ⇒ *real* ($*s*′-- *200*)
where $*s*-*n* ≡ $*s*^{*1*}-*n*

definition *ann-due* :: *nat* ⇒ *nat* ⇒ *real* ($*a*′′′′^{-}′-- [*0*,*101*] *200*)
where $*a*′′^{*m*}-*n* ≡ ∑ *k*<*n*∗*m*. $*v*.^((*k*::*nat*)/*m*) / *m*
— present value of an annuity-due

abbreviation *ann-due-yr* :: *nat* ⇒ *real* ($*a*′′′′′-- [*101*] *200*)
where $*a*′′-*n* ≡ $*a*′′^{*1*}-*n*

definition *acc-due* :: *nat* ⇒ *nat* ⇒ *real* ($*s*′′′′^{-}′-- [*0*,*101*] *200*)
where $*s*′′^{*m*}-*n* ≡ ∑ *k*<*n*∗*m*. (*1*+*i*).^((*k*+*1*::*nat*)/*m*) / *m*
— future value of an annuity-due

abbreviation *acc-due-yr* :: *nat* ⇒ *real* ($*s*′′′′′-- [*101*] *200*)
where $*s*′′-*n* ≡ $*s*′′^{*1*}-*n*

definition *ann-cont* :: *real* ⇒ *real* ($*a*′′′-- [*101*] *200*)
where $*a*′-*n* ≡ *integral* {*0*..*n*} (λ*t*::*real*. $*v*.^*t*)
— present value of a continuous annuity

definition *acc-cont* :: *real* ⇒ *real* ($*s*′′′-- [*101*] *200*)
where $*s*′-*n* ≡ *integral* {*0*..*n*} (λ*t*::*real*. (*1*+*i*).^*t*)
— future value of a continuous annuity

definition *perp* :: *nat* ⇒ *real* ($*a*^{-}′-∞ [*0*] *200*)
where $*a*^{*m*}-∞ ≡ *1* / $*i*^{*m*}
— present value of a perpetual annuity

abbreviation *perp-yr* :: *real* ($*a*′-∞ *200*)
where $*a*-∞ ≡ $*a*^{*1*}-∞

definition *perp-due* :: *nat* ⇒ *real* ($*a*′′′′^{-}′-∞ [*0*] *200*)
where $*a*′′^{*m*}-∞ ≡ *1* / $*d*^{*m*}
— present value of a perpetual annuity-due

abbreviation *perp-due-yr* :: *real* ($\$a''''$-$\infty$ *200*)
  where $\$a''$-$\infty$ $\equiv$ $\$a''$^{*1*}-$\infty$

definition *ann-incr* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *real* ($'(I^{-}a')^{-}'$-- [*0*,*0*,*101*] *200*)
  where $\$(I^{*l*}a)$^{*m*}-*n* $\equiv$ $\sum$ *k*<*n*∗*m*. $v.$^((*k*+*1*::*nat*)/*m*) ∗ ⌈*l*∗(*k*+*1*::*nat*)/*m*⌉ / (*l*∗*m*)
  — present value of an increasing annuity
  — This is my original definition.
  — Here, *l* represents the number of increments per unit time.

abbreviation *ann-incr-lvl* :: *nat* ⇒ *nat* ⇒ *real* ($'(Ia')^{-}'$-- [*0*,*101*] *200*)
  where (Ia)^{*m*}-*n* $\equiv$ $(I^{*1*}a)$^{*m*}-*n*
  — The post-fix *lvl* stands for "level".

abbreviation *ann-incr-yr* :: *nat* ⇒ *real* ($'(Ia')'$-- [*101*] *200*)
  where (Ia)-*n* $\equiv$ (Ia)^{*1*}-*n*

definition *acc-incr* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *real* ($'(I^{-}s')^{-}'$-- [*0*,*0*,*101*] *200*)
  where $(I^{*l*}s)$^{*m*}-*n* $\equiv$ $\sum$ *k*<*n*∗*m*. (*1*+*i*).^(*n*−(*k*+*1*::*nat*)/*m*) ∗ ⌈*l*∗(*k*+*1*::*nat*)/*m*⌉ / (*l*∗*m*)
  — future value of an increasing annuity

abbreviation *acc-incr-lvl* :: *nat* ⇒ *nat* ⇒ *real* ($'(Is')^{-}'$-- [*0*,*101*] *200*)
  where (Is)^{*m*}-*n* $\equiv$ $(I^{*1*}s)$^{*m*}-*n*

abbreviation *acc-incr-yr* :: *nat* ⇒ *real* ($'(Is')'$-- [*101*] *200*)
  where (Is)-*n* $\equiv$ (Is)^{*1*}-*n*

definition *ann-due-incr* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *real* ($'(I^{-}a'''')^{-}'$-- [*0*,*0*,*101*] *200*)
  where $(I^{*l*}a'')$^{*m*}-*n* $\equiv$ $\sum$ *k*<*n*∗*m*. $v.$^((*k*::*nat*)/*m*) ∗ ⌈*l*∗(*k*+*1*::*nat*)/*m*⌉ / (*l*∗*m*)

abbreviation *ann-due-incr-lvl* :: *nat* ⇒ *nat* ⇒ *real* ($'(Ia'''')^{-}'$-- [*0*,*101*] *200*)
  where (Ia'')^{*m*}-*n* $\equiv$ $(I^{*1*}a'')$^{*m*}-*n*

abbreviation *ann-due-incr-yr* :: *nat* ⇒ *real* ($'(Ia'''')'$-- [*101*] *200*)
  where (Ia'')-*n* $\equiv$ (Ia'')^{*1*}-*n*

definition *acc-due-incr* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *real* ($'(I^{-}s'''')^{-}'$-- [*0*,*0*,*101*] *200*)
  where $(I^{*l*}s'')$^{*m*}-*n* $\equiv$ $\sum$ *k*<*n*∗*m*. (*1*+*i*).^(*n*−(*k*::*nat*)/*m*) ∗ ⌈*l*∗(*k*+*1*::*nat*)/*m*⌉ / (*l*∗*m*)

abbreviation *acc-due-incr-lvl* :: *nat* ⇒ *nat* ⇒ *real* ($'(Is'''')^{-}'$-- [*0*,*101*] *200*)
  where (Is'')^{*m*}-*n* $\equiv$ $(I^{*1*}s'')$^{*m*}-*n*

abbreviation *acc-due-incr-yr* :: *nat* ⇒ *real* ($'(Is'''')'$-- [*101*] *200*)
  where (Is'')-*n* $\equiv$ (Is'')^{*1*}-*n*

definition $\textit{perp-incr} :: \textit{nat} \Rightarrow \textit{nat} \Rightarrow \textit{real}$ ($\$'(I\hat{}\{\text{-}\}a')\hat{}\{\text{-}\}'\text{-}\infty$ [0,0] 200)
where $\$(I\hat{}\{l\}a)\hat{}\{m\}\text{-}\infty \equiv \textit{lim}\ (\lambda n.\ \$(I\hat{}\{l\}a)\hat{}\{m\}\text{-}n)$

abbreviation $\textit{perp-incr-lvl} :: \textit{nat} \Rightarrow \textit{real}$ ($\$'(Ia')\hat{}\{\text{-}\}'\text{-}\infty$ [0] 200)
where $\$(Ia)\hat{}\{m\}\text{-}\infty \equiv \$(I\hat{}\{1\}a)\hat{}\{m\}\text{-}\infty$

abbreviation $\textit{perp-incr-yr} :: \textit{real}$ ($\$'(Ia')'\text{-}\infty$ 200)
where $\$(Ia)\text{-}\infty \equiv \$(Ia)\hat{}\{1\}\text{-}\infty$

definition $\textit{perp-due-incr} :: \textit{nat} \Rightarrow \textit{nat} \Rightarrow \textit{real}$ ($\$'(I\hat{}\{\text{-}\}a'''')\hat{}\{\text{-}\}'\text{-}\infty$ [0,0] 200)
where $\$(I\hat{}\{l\}a'')\hat{}\{m\}\text{-}\infty \equiv \textit{lim}\ (\lambda n.\ \$(I\hat{}\{l\}a'')\hat{}\{m\}\text{-}n)$

abbreviation $\textit{perp-due-incr-lvl} :: \textit{nat} \Rightarrow \textit{real}$ ($\$'(Ia'''')\hat{}\{\text{-}\}'\text{-}\infty$ [0] 200)
where $\$(Ia'')\hat{}\{m\}\text{-}\infty \equiv \$(I\hat{}\{1\}a'')\hat{}\{m\}\text{-}\infty$

abbreviation $\textit{perp-due-incr-yr} :: \textit{real}$ ($\$'(Ia'''')'\text{-}\infty$ 200)
where $\$(Ia'')\text{-}\infty \equiv \$(Ia'')\hat{}\{1\}\text{-}\infty$

lemma *v-futr-m-pos*: $1 + \$i\hat{}\{m\}/m > 0$ **if** $m \neq 0$ **for** $m::\textit{nat}$
⟨*proof*⟩

lemma *i-nom-1*[*simp*]: $\$i\hat{}\{1\} = i$
⟨*proof*⟩

lemma *i-nom-eff*: $(1 + \$i\hat{}\{m\}/m)\hat{}m = 1 + i$ **if** $m \neq 0$ **for** $m::\textit{nat}$
⟨*proof*⟩

lemma *i-nom-i*: $1 + \$i\hat{}\{m\}/m = (1+i).\hat{}(1/m)$ **if** $m \neq 0$ **for** $m::\textit{nat}$
⟨*proof*⟩

lemma *i-nom-0-iff-i-0*: $\$i\hat{}\{m\} = 0 \longleftrightarrow i = 0$ **if** $m \neq 0$ **for** $m::\textit{nat}$
⟨*proof*⟩

lemma *i-nom-pos-iff-i-pos*: $\$i\hat{}\{m\} > 0 \longleftrightarrow i > 0$ **if** $m \neq 0$ **for** $m::\textit{nat}$
⟨*proof*⟩

lemma *e-delta*: $\textit{exp}\ (\$\delta) = 1 + i$
⟨*proof*⟩

lemma *delta-0-iff-i-0*: $\$\delta = 0 \longleftrightarrow i = 0$
⟨*proof*⟩

lemma *lim-i-nom*: $(\lambda m.\ \$i\hat{}\{m\}) \longlonglongrightarrow \$\delta$
⟨*proof*⟩

lemma *d-nom-0-iff-i-0*: $\$d\hat{}\{m\} = 0 \longleftrightarrow i = 0$ **if** $m \neq 0$ **for** $m::\textit{nat}$
⟨*proof*⟩

lemma *d-nom-pos-iff-i-pos*: $d^{m} > 0 ⟷ i > 0 **if** m ≠ 0 **for** m::nat
⟨*proof*⟩

lemma *d-nom-i-nom*: 1 − $d^{m}/m = 1 / (1 + $i^{m}/m) **if** m ≠ 0 **for** m::nat
⟨*proof*⟩

lemma *lim-d-nom*: (λm. $d^{m}) ⟶ $δ
⟨*proof*⟩

lemma *v-pos*: $v > 0
⟨*proof*⟩

lemma *v-1-iff-i-0*: $v = 1 ⟷ i = 0
⟨*proof*⟩

lemma *v-lt-1-iff-i-pos*: $v < 1 ⟷ i > 0
⟨*proof*⟩

lemma *v-i-nom*: $v = (1 + $i^{m}/m).^−m **if** m ≠ 0 **for** m::nat
⟨*proof*⟩

lemma *i-v*: 1 + i = $v.^−1
⟨*proof*⟩

lemma *i-v-powr*: (1 + i).^a = $v.^−a **for** a::real
⟨*proof*⟩

lemma *v-delta*: ln ($v) = − $δ
⟨*proof*⟩

lemma *is-derive-vpow*: DERIV (λt. $v.^t) t :> − $δ ∗ $v.^t
⟨*proof*⟩

lemma *d-nom-v*: $d^{m} = m ∗ (1 − $v.^(1/m)) **if** m ≠ 0 **for** m::nat
⟨*proof*⟩

lemma *d-nom-i-nom-v*: $d^{m} = $i^{m} ∗ $v.^(1/m) **if** m ≠0 **for** m::nat
⟨*proof*⟩

lemma *a-calc*: $a^{m}-n = (1 − $v^n) / $i^{m} **if** m ≠ 0 i ≠ 0 **for** n m ::nat
⟨*proof*⟩

lemma *a-calc-i-0*: $a^{m}-n = n **if** m ≠ 0 i = 0 **for** n m :: nat
⟨*proof*⟩

lemma *s-calc-i-0*: $s^{m}-n = n **if** m ≠ 0 i = 0 **for** n m :: nat
⟨*proof*⟩

lemma *s-a*: $s^{m}-n = (1+i)^n ∗ $a^{m}-n **if** m ≠ 0 **for** n m :: nat

⟨*proof*⟩

lemma *s-calc*: $\$s\hat{}\{m\}\text{-}n = ((1+i)\hat{}n - 1)\ /\ \$i\hat{}\{m\}$ **if** $m \neq 0$ $i \neq 0$ **for** $n\ m :: nat$
⟨*proof*⟩

lemma *a''-a*: $\$a''\hat{}\{m\}\text{-}n = (1+i).\hat{}(1/m) * \$a\hat{}\{m\}\text{-}n$ **if** $m \neq 0$ **for** $m{::}nat$
⟨*proof*⟩

lemma *a-a''*: $\$a\hat{}\{m\}\text{-}n = \$v.\hat{}(1/m) * \$a''\hat{}\{m\}\text{-}n$ **if** $m \neq 0$ **for** $m{::}nat$
⟨*proof*⟩

lemma *a''-calc-i-0*: $\$a''\hat{}\{m\}\text{-}n = n$ **if** $m \neq 0$ $i = 0$ **for** $n\ m :: nat$
⟨*proof*⟩

lemma *s''-calc-i-0*: $\$s''\hat{}\{m\}\text{-}n = n$ **if** $m \neq 0$ $i = 0$ **for** $n\ m :: nat$
⟨*proof*⟩

lemma *a''-calc*: $\$a''\hat{}\{m\}\text{-}n = (1 - \$v\hat{}n)\ /\ \$d\hat{}\{m\}$ **if** $m \neq 0$ $i \neq 0$ **for** $n\ m :: nat$
⟨*proof*⟩

lemma *s''-s*: $\$s''\hat{}\{m\}\text{-}n = (1+i).\hat{}(1/m) * \$s\hat{}\{m\}\text{-}n$ **if** $m \neq 0$ **for** $m{::}nat$
⟨*proof*⟩

lemma *s-s''*: $\$s\hat{}\{m\}\text{-}n = \$v.\hat{}(1/m) * \$s''\hat{}\{m\}\text{-}n$ **if** $m \neq 0$ **for** $m{::}nat$
⟨*proof*⟩

lemma *s''-calc*: $\$s''\hat{}\{m\}\text{-}n = ((1+i)\hat{}n - 1)\ /\ \$d\hat{}\{m\}$ **if** $m \neq 0$ $i \neq 0$ **for** $n\ m :: nat$
⟨*proof*⟩

lemma *s''-a''*: $\$s''\hat{}\{m\}\text{-}n = (1+i)\hat{}n * \$a''\hat{}\{m\}\text{-}n$ **if** $m \neq 0$ **for** $m{::}nat$
⟨*proof*⟩

lemma *a'-calc*: $\$a'\text{-}n = (1 - \$v.\hat{}n)\ /\ \$\delta$ **if** $i \neq 0$ $n \geq 0$ **for** $n{::}real$
⟨*proof*⟩

lemma *a'-calc-i-0*: $\$a'\text{-}n = n$ **if** $i = 0$ $n \geq 0$ **for** $n{::}real$
⟨*proof*⟩

lemma *s'-calc*: $\$s'\text{-}n = ((1+i).\hat{}n - 1)\ /\ \$\delta$ **if** $i \neq 0$ $n \geq 0$ **for** $n{::}real$
⟨*proof*⟩

lemma *s'-calc-i-0*: $\$s'\text{-}n = n$ **if** $i = 0$ $n \geq 0$ **for** $n{::}real$
⟨*proof*⟩

lemma *s'-a'*: $\$s'\text{-}n = (1+i).\hat{}n * \$a'\text{-}n$ **if** $n \geq 0$ **for** $n{::}real$
⟨*proof*⟩

lemma *lim-m-a*: $(\lambda m.\ \$a\hat{}\{m\}\text{-}n) \longlonglongrightarrow \$a'\text{-}n$ **for** *n::nat*
⟨*proof*⟩

lemma *lim-m-a''*: $(\lambda m.\ \$a''\hat{}\{m\}\text{-}n) \longlonglongrightarrow \$a'\text{-}n$ **for** *n::nat*
⟨*proof*⟩

lemma *lim-m-s*: $(\lambda m.\ \$s\hat{}\{m\}\text{-}n) \longlonglongrightarrow \$s'\text{-}n$ **for** *n::nat*
⟨*proof*⟩

lemma *lim-m-s''*: $(\lambda m.\ \$s''\hat{}\{m\}\text{-}n) \longlonglongrightarrow \$s'\text{-}n$ **for** *n::nat*
⟨*proof*⟩

lemma *lim-n-a*: $(\lambda n.\ \$a\hat{}\{m\}\text{-}n) \longlonglongrightarrow \$a\hat{}\{m\}\text{-}\infty$ **if** $m \neq 0$ $i > 0$ **for** *m::nat*
⟨*proof*⟩

lemma *lim-n-a''*: $(\lambda n.\ \$a''\hat{}\{m\}\text{-}n) \longlonglongrightarrow \$a''\hat{}\{m\}\text{-}\infty$ **if** $m \neq 0$ $i > 0$ **for** *m::nat*
⟨*proof*⟩

lemma *Ilsm-Ilam*: $\$(I\hat{}\{l\}s)\hat{}\{m\}\text{-}n = (1+i)\hat{}n * \$(I\hat{}\{l\}a)\hat{}\{m\}\text{-}n$
  if $l \neq 0$ $m \neq 0$ **for** *l n m :: nat*
  ⟨*proof*⟩

lemma *Iam-calc*: $\$(Ia)\hat{}\{m\}\text{-}n = (\sum j{<}n.\ (j+1)/m * (\sum k{=}j{*}m..{<}(j+1){*}m.\ \$v.\hat{}((k+1)/m)))$
  if $m \neq 0$ **for** *n m :: nat*
⟨*proof*⟩

lemma *Ism-calc*: $\$(Is)\hat{}\{m\}\text{-}n = (\sum j{<}n.\ (j+1)/m * (\sum k{=}j{*}m..{<}(j+1){*}m.\ (1+i).\hat{}(n-(k+1)/m)))$
  if $m \neq 0$ **for** *n m :: nat*
  ⟨*proof*⟩

lemma *Imam-calc-aux*: $\$(I\hat{}\{m\}a)\hat{}\{m\}\text{-}n = (\sum k{<}n{*}m.\ \$v.\hat{}((k+1)/m) * (k+1)$
$/\ m\hat{}2)$
  if $m \neq 0$ **for** *m::nat*
  ⟨*proof*⟩

lemma *Imam-calc*:
  $\$(I\hat{}\{m\}a)\hat{}\{m\}\text{-}n = (\$v.\hat{}(1/m) * (1 - (n{*}m+1){*}\$v\hat{}n + n{*}m{*}\$v.\hat{}(n+1/m)))$
$/\ (m{*}(1-\$v.\hat{}(1/m)))\hat{}2$
  if $i \neq 0$ $m \neq 0$ **for** *n m :: nat*
⟨*proof*⟩

lemma *Imam-calc-i-0*: $\$(I\hat{}\{m\}a)\hat{}\{m\}\text{-}n = (n{*}m+1){*}n\ /\ (2{*}m)$ **if** $i = 0$ $m \neq 0$
for *n m :: nat*
⟨*proof*⟩

lemma *Imsm-calc*:
  $\$(I\hat{}\{m\}s)\hat{}\{m\}\text{-}n = ((1+i).\hat{}(n+1/m) - (n{*}m+1){*}(1+i).\hat{}(1/m) + n{*}m)\ /$
$(m{*}((1+i).\hat{}(1/m)-1))\hat{}2$

if $i \neq 0$ $m \neq 0$ **for** *n m* :: *nat*
⟨*proof*⟩

lemma *Imsm-calc-i-0*: $(I^{m}s)^{m}-n = (n*m+1)*n / (2*m) **if** $i = 0$ $m \neq 0$ **for** *n m* :: *nat*
⟨*proof*⟩

lemma *Ila′′m-Ilam*: $(I^{l}a′′)^{m}-n = (1+i).^(1/m) * $(I^{l}a)^{m}-n
if $l \neq 0$ $m \neq 0$ **for** *l m n* :: *nat*
⟨*proof*⟩

lemma *Ia′′m-calc*: $(Ia′′)^{m}-n = (∑j<n. (j+1)/m * (∑k=j*m..<(j+1)*m. $v.^(k/m)))
if $m \neq 0$ **for** *n m* :: *nat*
⟨*proof*⟩

lemma *Ima′′m-calc-aux*: $(I^{m}a′′)^{m}-n = (∑k<n*m. $v.^(k/m) * (k+1) / m^2)
if $m \neq 0$ **for** *m*::*nat*
⟨*proof*⟩

lemma *Ima′′m-calc*: $(I^{m}a′′)^{m}-n = (1 − (n*m+1)*$v^n + n*m*$v.^(n+1/m)) / (m*(1−$v.^(1/m)))^2
if $i \neq 0$ $m \neq 0$ **for** *n m* :: *nat*
⟨*proof*⟩

lemma *Ils′′m-Ilsm*: $(I^{l}s′′)^{m}-n = (1+i).^(1/m) * $(I^{l}s)^{m}-n
if $l \neq 0$ $m \neq 0$ **for** *l m n* :: *nat*
⟨*proof*⟩

lemma *Ims′′m-calc*:
$(I^{m}s′′)^{m}-n =
(1+i).^(1/m) * ((1+i).^(n+1/m) − (n*m+1)*(1+i).^(1/m) + n*m) / (m*((1+i).^(1/m)−1))^2
if $i \neq 0$ $m \neq 0$ **for** *n m* :: *nat*
⟨*proof*⟩

lemma *lim-Imam*: (λn. $(I^{m}a)^{m}-n) ⟶ 1 / ($i^{m}*$d^{m}) **if** $m \neq 0$ $i > 0$ **for** *m*::*nat*
⟨*proof*⟩

lemma *perp-incr-calc*: $(I^{m}a)^{m}-∞ = 1 / ($i^{m}*$d^{m}) **if** $m \neq 0$ $i > 0$ **for** *m*::*nat*
⟨*proof*⟩

lemma *lim-Ima′′m*: (λn. $(I^{m}a′′)^{m}-n) ⟶ 1 / ($d^{m})^2 **if** $m \neq 0$ $i > 0$ **for** *m*::*nat*
⟨*proof*⟩

lemma *perp-due-incr-calc*: $\$(I\hat{}\{m\}a'')\hat{}\{m\}\text{-}\infty = 1\ /\ (\$d\hat{}\{m\})\hat{}2$ **if** $m \neq 0$ $i > 0$ **for** *m*::*nat*
⟨*proof*⟩

end

end
theory *Survival-Model*
imports *HOL−Library.Rewrite HOL−Library.Extended-Nonnegative-Real HOL−Library.Extended-Real HOL−Probability.Probability Preliminaries*
begin

5 Survival Model

The survival model is built on the probability space $\mathfrak{M}$. Additionally, the random variable X : *space* $\mathfrak{M} \to \mathbb{R}$ is introduced, which represents the age at death.

locale *prob-space-actuary* = *MM-PS*: *prob-space* $\mathfrak{M}$ **for** $\mathfrak{M}$
— Since the letter M may be used as a commutation function, adopt the letter $\mathfrak{M}$ instead as a notation for the measure space.

locale *survival-model* = *prob-space-actuary* +
fixes X :: *'a* ⇒ *real*
assumes *X-RV*[*simp*]: *MM-PS.random-variable* (*borel* :: *real measure*) *X*
and *X-pos-AE*[*simp*]: *AE* ξ *in* $\mathfrak{M}$. $X\ \xi > 0$
begin

5.1 General Theory of Survival Model

interpretation *distrX-RD*: *real-distribution distr* $\mathfrak{M}$ *borel X*
⟨*proof*⟩

lemma *X-le-event*[*simp*]: $\{\xi \in$ *space* $\mathfrak{M}.\ X\ \xi \le x\} \in$ *MM-PS.events*
⟨*proof*⟩

lemma *X-gt-event*[*simp*]: $\{\xi \in$ *space* $\mathfrak{M}.\ X\ \xi > x\} \in$ *MM-PS.events*
⟨*proof*⟩

lemma *X-compl-le-gt*: *space* $\mathfrak{M} - \{\xi \in$ *space* $\mathfrak{M}.\ X\ \xi \le x\} = \{\xi \in$ *space* $\mathfrak{M}.\ X\ \xi > x\}$ **for** *x*::*real*
⟨*proof*⟩

lemma *X-compl-gt-le*: *space* $\mathfrak{M} - \{\xi \in$ *space* $\mathfrak{M}.\ X\ \xi > x\} = \{\xi \in$ *space* $\mathfrak{M}.\ X\ \xi \le x\}$ **for** *x*::*real*
⟨*proof*⟩

5.1.1 Introduction of Survival Function for X

Note that *ccdf* (*distr* $\mathfrak{M}$ *borel* X) is the survival (distributive) function for X.

lemma *ccdfX-0-1*: *ccdf* (*distr* $\mathfrak{M}$ *borel* X) *0* = *1*
⟨*proof*⟩

lemma *ccdfX-unborn-1*: *ccdf* (*distr* $\mathfrak{M}$ *borel* X) x = *1* **if** $x \leq 0$
⟨*proof*⟩

definition *death-pt* :: *ereal* ($\$\psi$)
where $\$\psi \equiv$ *Inf* (*ereal* ‘ $\{x \in \mathbb{R}$. *ccdf* (*distr* $\mathfrak{M}$ *borel* X) $x = 0\}$)
— This is my original notation, which is used to develop life insurance mathematics rigorously.

lemma *psi-nonneg*: $\$\psi \geq 0$
⟨*proof*⟩

lemma *ccdfX-beyond-0*: *ccdf* (*distr* $\mathfrak{M}$ *borel* X) x = *0* **if** $x > \$\psi$ **for** x::*real*
⟨*proof*⟩

lemma *ccdfX-psi-0*: *ccdf* (*distr* $\mathfrak{M}$ *borel* X) (*real-of-ereal* $\$\psi$) = *0* **if** $\$\psi < \infty$
⟨*proof*⟩

lemma *ccdfX-0-equiv*: *ccdf* (*distr* $\mathfrak{M}$ *borel* X) $x = 0 \longleftrightarrow x \geq \$\psi$ **for** x::*real*
⟨*proof*⟩

lemma *psi-pos*[*simp*]: $\$\psi > 0$
⟨*proof*⟩

corollary *psi-pos'*[*simp*]: $\$\psi >$ *ereal 0*
⟨*proof*⟩

5.1.2 Introdution of Future-Lifetime Random Variable $T(x)$

definition *alive* :: *real* ⇒ *'a set*
where *alive* $x \equiv \{\xi \in$ *space* $\mathfrak{M}$. $X\ \xi > x\}$

lemma *alive-event*[*simp*]: *alive* $x \in$ *MM-PS.events* **for** x::*real*
⟨*proof*⟩

lemma *X-alivex-measurable*[*measurable*, *simp*]: $X \in$ *borel-measurable* ($\mathfrak{M}$ ⇂ *alive* x) **for** x::*real*
⟨*proof*⟩

definition *futr-life* :: *real* ⇒ (*'a* ⇒ *real*) (T)
where $T\ x \equiv (\lambda\xi.\ X\ \xi - x)$
— Note that $T(x)$: *space* $\mathfrak{M} \to \mathbb{R}$ represents the time until death of a person aged x.

lemma *T0-eq-X*[*simp*]: *T 0* = *X*
⟨*proof*⟩

lemma *Tx-measurable*[*measurable*, *simp*]: *T x* ∈ *borel-measurable* 𝔐 **for** *x*::*real*
⟨*proof*⟩

lemma *Tx-alivex-measurable*[*measurable*, *simp*]: *T x* ∈ *borel-measurable* (𝔐 ⇂ *alive x*) **for** *x*::*real*
⟨*proof*⟩

lemma *alive-T*: *alive x* = {ξ ∈ *space* 𝔐. *T x* ξ > *0*} **for** *x*::*real*
⟨*proof*⟩

lemma *alivex-Tx-pos*[*simp*]: *0* < *T x* ξ **if** ξ ∈ *space* (𝔐 ⇂ *alive x*) **for** *x*::*real*
⟨*proof*⟩

lemma *PT0-eq-PX-lborel*: 𝒫(ξ *in* 𝔐. *T 0* ξ ∈ *A* | *T 0* ξ > *0*) = 𝒫(ξ *in* 𝔐. *X* ξ ∈ *A*)
if *A* ∈ *sets lborel* **for** *A* :: *real set*
⟨*proof*⟩

5.1.3 Actuarial Notations on the Survival Model

definition *survive* :: *real* ⇒ *real* ⇒ *real* (\$*p*′-{-&-} [*0*,*0*] *200*)
where \$*p*-{*t*&*x*} ≡ *ccdf* (*distr* (𝔐 ⇂ *alive x*) *borel* (*T x*)) *t*
— the probability that a person aged *x* will survive for *t* years
— Note that the function \$*p*-{·&*x*} is the survival function on (𝔐 ⇂ *alive x*) for the random variable *T*(*x*).
— The parameter *t* is usually nonnegative, but theoretically it can be negative.

abbreviation *survive-1* :: *real* ⇒ *real* (\$*p*′-- [*101*] *200*)
where \$*p*-*x* ≡ \$*p*-{*1*&*x*}

definition *die* :: *real* ⇒ *real* ⇒ *real* (\$*q*′-{-&-} [*0*,*0*] *200*)
where \$*q*-{*t*&*x*} ≡ *cdf* (*distr* (𝔐 ⇂ *alive x*) *borel* (*T x*)) *t*
— the probability that a person aged *x* will die within *t* years
— Note that the function \$*q*-{·&*x*} is the cumulative distributive function on (𝔐 ⇂ *alive x*) for the random variable *T*(*x*).
— The parameter *t* is usually nonnegative, but theoretically it can be negative.

abbreviation *die-1* :: *real* ⇒ *real* (\$*q*′-- [*101*] *200*)
where \$*q*-*x* ≡ \$*q*-{*1*&*x*}

definition *die-defer* :: *real* ⇒ *real* ⇒ *real* ⇒ *real* (\$*q*′-{-|-&-} [*0*,*0*,*0*] *200*)
where \$*q*-{*f*|*t*&*x*} = |\$*q*-{*f*+*t*&*x*} − \$*q*-{*f*&*x*}|
— the probability that a person aged *x* will die within *t* years, deferred *f* years
— The parameters *f* and *t* are usually nonnegative, but theoretically they can be negative.

abbreviation *die-defer-1* :: *real* ⇒ *real* ⇒ *real* (\$*q*′-{-|&-} [*0*,*0*] *200*)
where \$*q*-{*f*|&*x*} ≡ \$*q*-{*f*|*1*&*x*}

definition *life-expect* :: *real* ⇒ *real* (\$*e*'○'-- [*101*] *200*)
where \$*e*'○-*x* ≡ $integral^L$ (𝔐 ⌊ *alive x*) (*T x*)
— complete life expectation
— Note that \$*e*'○-*x* is calculated as *0* when *nn-integral* (𝔐 ⌊ *alve x*) (*T x*) = ∞.

definition *temp-life-expect* :: *real* ⇒ *real* ⇒*real* (\$*e*'○′-{-:-} [*0*,*0*] *200*)
where \$*e*'○-{*x*:*n*} ≡ $integral^L$ (𝔐 ⌊ *alive x*) (λξ. *min* (*T x ξ*) *n*)
— temporary complete life expectation

definition *curt-life-expect* :: *real* ⇒ *real* (\$*e*′-- [*101*] *200*)
where \$*e*-*x* ≡ $integral^L$ (𝔐 ⌊ *alive x*) (λξ. ⌊*T x ξ*⌋)
— curtate life expectation
— Note that \$*e*-*x* is calculated as *0* when *nn-integral* (𝔐 ⌊ *alive x*) ⌊*T x*⌋ = ∞.

definition *temp-curt-life-expect* :: *real* ⇒ *real* ⇒ *real* (\$*e*′-{-:-} [*0*,*0*] *200*)
where \$*e*-{*x*:*n*} ≡ $integral^L$ (𝔐 ⌊ *alive x*) (λξ. ⌊*min* (*T x ξ*) *n*⌋)
— temporary curtate life expectation
— In the definition *n* can be a real number, but in practice *n* is usually a natural number.

5.1.4 Properties of Survival Function for $T(x)$

context
fixes *x*::*real*
assumes *x-lt-psi*[*simp*]: *x* < \$*ψ*
begin

lemma *PXx-pos*[*simp*]: 𝒫(*ξ in* 𝔐. *X ξ* > *x*) > *0*
⟨*proof*⟩

lemma *PTx-pos*[*simp*]: 𝒫(*ξ in* 𝔐. *T x ξ* > *0*) > *0*
⟨*proof*⟩

interpretation *alivex-PS*: *prob-space* 𝔐 ⌊ *alive x*
⟨*proof*⟩

interpretation *distrTx-RD*: *real-distribution distr* (𝔐 ⌊ *alive x*) *borel* (*T x*) ⟨*proof*⟩

lemma *ccdfTx-cond-prob*:
ccdf (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *t* = 𝒫(*ξ in* 𝔐. *T x ξ* > *t* | *T x ξ* > *0*) **for** *t*::*real*
⟨*proof*⟩

lemma *ccdfTx-0-1*: *ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *0* = *1*

⟨*proof*⟩

lemma *ccdfTx-nonpos-1*: *ccdf* (*distr* ($\mathfrak{M}$ ↓ *alive x*) *borel* (*T x*)) *t* = *1* **if** *t* ≤ *0* **for** *t* :: *real*
⟨*proof*⟩

lemma *ccdfTx-0-equiv*: *ccdf* (*distr* ($\mathfrak{M}$ ↓ *alive x*) *borel* (*T x*)) *t* = *0* ⟷ *x*+*t* ≥ \$ψ **for** *t*::*real*
⟨*proof*⟩

lemma *ccdfTx-continuous-on-nonpos*[*simp*]:
continuous-on {..*0*} (*ccdf* (*distr* ($\mathfrak{M}$ ↓ *alive x*) *borel* (*T x*)))
⟨*proof*⟩

lemma *ccdfTx-differentiable-on-nonpos*[*simp*]:
(*ccdf* (*distr* ($\mathfrak{M}$ ↓ *alive x*) *borel* (*T x*))) *differentiable-on* {..*0*}
⟨*proof*⟩

lemma *ccdfTx-has-real-derivative-0-at-neg*:
(*ccdf* (*distr* ($\mathfrak{M}$ ↓ *alive x*) *borel* (*T x*)) *has-real-derivative 0*) (*at t*) **if** *t* < *0* **for** *t*::*real*
⟨*proof*⟩

lemma *ccdfTx-integrable-Icc*:
set-integrable lborel {*a*..*b*} (*ccdf* (*distr* ($\mathfrak{M}$ ↓ *alive x*) *borel* (*T x*))) **for** *a b* :: *real*
⟨*proof*⟩

corollary *ccdfTx-integrable-on-Icc*:
ccdf (*distr* ($\mathfrak{M}$ ↓ *alive x*) *borel* (*T x*)) *integrable-on* {*a*..*b*} **for** *a b* :: *real*
⟨*proof*⟩

lemma *ccdfTx-PX*:
ccdf (*distr* ($\mathfrak{M}$ ↓ *alive x*) *borel* (*T x*)) *t* = $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *X* ξ > *x*+*t*) / $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *X* ξ > *x*)
if *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *ccdfTx-ccdfX*: *ccdf* (*distr* ($\mathfrak{M}$ ↓ *alive x*) *borel* (*T x*)) *t* =
ccdf (*distr* $\mathfrak{M}$ *borel X*) (*x* + *t*) / *ccdf* (*distr* $\mathfrak{M}$ *borel X*) *x*
if *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *ccdfT0-eq-ccdfX*: *ccdf* (*distr* ($\mathfrak{M}$ ↓ *alive 0*) *borel* (*T 0*)) = *ccdf* (*distr* $\mathfrak{M}$ *borel X*)
⟨*proof*⟩

lemma *continuous-ccdfX-ccdfTx*:
continuous (*at* (*x*+*t*) *within* {*x*..}) (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) ⟷
continuous (*at t within* {*0*..}) (*ccdf* (*distr* ($\mathfrak{M}$ ↓ *alive x*) *borel* (*T x*)))

if $t \geq 0$ **for** *t::real*
⟨*proof*⟩

lemma *isCont-ccdfX-ccdfTx*:
isCont (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) (*x*+*t*) ⟷
isCont (*ccdf* (*distr* ($\mathfrak{M}$ ⇂ *alive x*) *borel* (*T x*))) *t*
if $t > 0$ **for** *t::real*
⟨*proof*⟩

lemma *has-real-derivative-ccdfX-ccdfTx*:
((*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) *has-real-derivative D*) (*at* (*x*+*t*)) ⟷
((*ccdf* (*distr* ($\mathfrak{M}$ ⇂ *alive x*) *borel* (*T x*))) *has-real-derivative* (*D* / $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *X* ξ > *x*))) (*at t*)
if $t > 0$ **for** *t D* :: *real*
⟨*proof*⟩

lemma *differentiable-ccdfX-ccdfTx*:
(*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) *differentiable at* (*x*+*t*) ⟷
(*ccdf* (*distr* ($\mathfrak{M}$ ⇂ *alive x*) *borel* (*T x*))) *differentiable at t*
if $t > 0$ **for** *t::real*
⟨*proof*⟩

5.1.5 Properties of \$*p*-{*t*&*x*}

lemma *p-0-1*: \$*p*-{*0*&*x*} = *1*
⟨*proof*⟩

lemma *p-nonneg*[*simp*]: \$*p*-{*t*&*x*} ≥ *0* **for** *t::real*
⟨*proof*⟩

lemma *p-le-1*[*simp*]: \$*p*-{*t*&*x*} ≤ *1* **for** *t::real*
⟨*proof*⟩

lemma *p-0-equiv*: \$*p*-{*t*&*x*} = *0* ⟷ *x*+*t* ≥ \$$\psi$ **for** *t::real*
⟨*proof*⟩

lemma *p-PTx*: \$*p*-{*t*&*x*} = $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *T x* ξ > *t* | *T x* ξ > *0*) **for** *t::real*
⟨*proof*⟩

lemma *p-PX*: \$*p*-{*t*&*x*} = $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *X* ξ > *x* + *t*) / $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *X* ξ > *x*) **if** *t* ≥ *0* **for** *t::real*
⟨*proof*⟩

lemma *p-mult*: \$*p*-{*t*+*t'* & *x*} = \$*p*-{*t*&*x*} ∗ \$*p*-{*t'* & *x*+*t*}
if *t* ≥ *0* *t'* ≥ *0* *x*+*t* < \$$\psi$ **for** *t t'* :: *real*
⟨*proof*⟩

lemma *p-PTx-ge-ccdf-isCont*: \$*p*-{*t*&*x*} = $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *T x* ξ ≥ *t* | *T x* ξ > *0*)
if *isCont* (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) (*x*+*t*) *t* > *0* **for** *t::real*

⟨*proof*⟩

end

5.1.6 Properties of Survival Function for X

lemma *ccdfX-continuous-unborn*[*simp*]: *continuous-on* {..*0*} (*ccdf* (*distr* $\mathfrak{M}$ *borel* *X*))
⟨*proof*⟩

lemma *ccdfX-differentiable-unborn*[*simp*]: (*ccdf* (*distr* $\mathfrak{M}$ *borel* *X*)) *differentiable-on* {..*0*}
⟨*proof*⟩

lemma *ccdfX-has-real-derivative-0-unborn*:
(*ccdf* (*distr* $\mathfrak{M}$ *borel* *X*) *has-real-derivative* *0*) (*at* *x*) **if** $x < 0$ **for** *x*::*real*
⟨*proof*⟩

lemma *ccdfX-integrable-Icc*:
set-integrable *lborel* {*a*..*b*} (*ccdf* (*distr* $\mathfrak{M}$ *borel* *X*)) **for** *a* *b* :: *real*
⟨*proof*⟩

corollary *ccdfX-integrable-on-Icc*:
ccdf (*distr* $\mathfrak{M}$ *borel* *X*) *integrable-on* {*a*..*b*} **for** *a* *b* :: *real*
⟨*proof*⟩

lemma *ccdfX-p*: *ccdf* (*distr* $\mathfrak{M}$ *borel* *X*) *x* = \$*p*-{*x*&*0*} **for** *x*::*real*
⟨*proof*⟩

5.1.7 Introduction of Cumulative Distributive Function for X

lemma *cdfX-0-0*: *cdf* (*distr* $\mathfrak{M}$ *borel* *X*) *0* = *0*
⟨*proof*⟩

lemma *cdfX-unborn-0*: *cdf* (*distr* $\mathfrak{M}$ *borel* *X*) *x* = *0* **if** $x \leq 0$
⟨*proof*⟩

lemma *cdfX-beyond-1*: *cdf* (*distr* $\mathfrak{M}$ *borel* *X*) *x* = *1* **if** $x > \$\psi$ **for** *x*::*real*
⟨*proof*⟩

lemma *cdfX-psi-1*: *cdf* (*distr* $\mathfrak{M}$ *borel* *X*) (*real-of-ereal* \$$\psi$) = *1* **if** $\$\psi < \infty$
⟨*proof*⟩

lemma *cdfX-1-equiv*: *cdf* (*distr* $\mathfrak{M}$ *borel* *X*) *x* = *1* $\longleftrightarrow$ $x \geq \$\psi$ **for** *x*::*real*
⟨*proof*⟩

5.1.8 Properties of Cumulative Distributive Function for $T(x)$

context
fixes *x*::*real*

assumes *x-lt-psi*[*simp*]: $x < \$\psi$
begin

interpretation *alivex-PS*: *prob-space* $\mathfrak{M} \downharpoonright$ *alive x*
⟨*proof*⟩

interpretation *distrTx-RD*: *real-distribution distr* ($\mathfrak{M} \downharpoonright$ *alive x*) *borel* (*T x*) ⟨*proof*⟩

lemma *cdfTx-cond-prob*:
cdf (*distr* ($\mathfrak{M} \downharpoonright$ *alive x*) *borel* (*T x*)) $t = \mathcal{P}(\xi$ *in* $\mathfrak{M}$. $T\ x\ \xi \le t \mid T\ x\ \xi > 0)$ **for** *t*::*real*
⟨*proof*⟩

lemma *cdfTx-0-0*: *cdf* (*distr* ($\mathfrak{M} \downharpoonright$ *alive x*) *borel* (*T x*)) *0* = *0*
⟨*proof*⟩

lemma *cdfTx-nonpos-0*: *cdf* (*distr* ($\mathfrak{M} \downharpoonright$ *alive x*) *borel* (*T x*)) *t* = *0* **if** $t \le 0$ **for** *t* :: *real*
⟨*proof*⟩

lemma *cdfTx-1-equiv*: *cdf* (*distr* ($\mathfrak{M} \downharpoonright$ *alive x*) *borel* (*T x*)) *t* = *1* $\longleftrightarrow x+t \ge \$\psi$ **for** *t*::*real*
⟨*proof*⟩

lemma *cdfTx-continuous-on-nonpos*[*simp*]:
continuous-on {..*0*} (*cdf* (*distr* ($\mathfrak{M} \downharpoonright$ *alive x*) *borel* (*T x*)))
⟨*proof*⟩

lemma *cdfTx-differentiable-on-nonpos*[*simp*]:
(*cdf* (*distr* ($\mathfrak{M} \downharpoonright$ *alive x*) *borel* (*T x*))) *differentiable-on* {..*0*}
⟨*proof*⟩

lemma *cdfTx-has-real-derivative-0-at-neg*:
(*cdf* (*distr* ($\mathfrak{M} \downharpoonright$ *alive x*) *borel* (*T x*)) *has-real-derivative 0*) (*at t*) **if** $t < 0$ **for** *t*::*real*
⟨*proof*⟩

lemma *cdfTx-integrable-Icc*:
set-integrable lborel {*a*..*b*} (*cdf* (*distr* ($\mathfrak{M} \downharpoonright$ *alive x*) *borel* (*T x*))) **for** *a b* :: *real*
⟨*proof*⟩

corollary *cdfTx-integrable-on-Icc*:
cdf (*distr* ($\mathfrak{M} \downharpoonright$ *alive x*) *borel* (*T x*)) *integrable-on* {*a*..*b*} **for** *a b* :: *real*
⟨*proof*⟩

lemma *cdfTx-PX*:
cdf (*distr* ($\mathfrak{M} \downharpoonright$ *alive x*) *borel* (*T x*)) $t = \mathcal{P}(\xi$ *in* $\mathfrak{M}$. $x < X\ \xi \wedge X\ \xi \le x+t) / \mathcal{P}(\xi$ *in* $\mathfrak{M}$. $X\ \xi > x)$
for *t*::*real*

⟨*proof*⟩

lemma *cdfT0-eq-cdfX*: *cdf* (*distr* ($\mathfrak{M}$ ↓ *alive 0*) *borel* (*T 0*)) = *cdf* (*distr* $\mathfrak{M}$ *borel* *X*)
⟨*proof*⟩

lemma *continuous-cdfX-cdfTx*:
continuous (*at* (*x*+*t*) *within* {*x*..}) (*cdf* (*distr* $\mathfrak{M}$ *borel* *X*)) ⟷
continuous (*at t within* {*0*..}) (*cdf* (*distr* ($\mathfrak{M}$ ↓ *alive x*) *borel* (*T x*)))
if $t \geq 0$ **for** *t*::*real*
⟨*proof*⟩

lemma *isCont-cdfX-cdfTx*:
isCont (*cdf* (*distr* $\mathfrak{M}$ *borel* *X*)) (*x*+*t*) ⟷
isCont (*cdf* (*distr* ($\mathfrak{M}$ ↓ *alive x*) *borel* (*T x*))) *t*
if $t > 0$ **for** *t*::*real*
⟨*proof*⟩

lemma *has-real-derivative-cdfX-cdfTx*:
((*cdf* (*distr* $\mathfrak{M}$ *borel* *X*)) *has-real-derivative D*) (*at* (*x*+*t*)) ⟷
((*cdf* (*distr* ($\mathfrak{M}$ ↓ *alive x*) *borel* (*T x*))) *has-real-derivative* (*D* / $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *X* ξ > *x*))) (*at t*)
if $t > 0$ **for** *t D* :: *real*
⟨*proof*⟩

lemma *differentiable-cdfX-cdfTx*:
(*cdf* (*distr* $\mathfrak{M}$ *borel* *X*)) *differentiable at* (*x*+*t*) ⟷
(*cdf* (*distr* ($\mathfrak{M}$ ↓ *alive x*) *borel* (*T x*))) *differentiable at t*
if $t > 0$ **for** *t*::*real*
⟨*proof*⟩

5.1.9 Properties of $q-{t&x}

lemma *q-nonpos-0*: $q-{*t*&*x*} = *0* **if** $t \leq 0$ **for** *t*::*real*
⟨*proof*⟩

corollary *q-0-0*: $q-{*0*&*x*} = *0*
⟨*proof*⟩

lemma *q-nonneg*[*simp*]: $q-{*t*&*x*} ≥ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *q-le-1*[*simp*]: $q-{*t*&*x*} ≤ *1* **for** *t*::*real*
⟨*proof*⟩

lemma *q-1-equiv*: $q-{*t*&*x*} = *1* ⟷ *x*+*t* ≥ $ψ **for** *t*::*real*
⟨*proof*⟩

lemma *q-PTx*: $q-{*t*&*x*} = $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *T x* ξ ≤ *t* | *T x* ξ > *0*) **for** *t*::*real*

⟨*proof*⟩

lemma *q-PX*: $\$q\text{-}\{t\&x\} = \mathcal{P}(\xi\ in\ \mathfrak{M}.\ x < X\ \xi \wedge X\ \xi \leq x + t)\ /\ \mathcal{P}(\xi\ in\ \mathfrak{M}.\ X\ \xi > x)$
⟨*proof*⟩

lemma *q-defer-0-q*[*simp*]: $\$q\text{-}\{0|t\&x\} = \$q\text{-}\{t\&x\}$ **for** *t*::*real*
⟨*proof*⟩

lemma *q-defer-0-0*: $\$q\text{-}\{f|0\&x\} = 0$ **for** *f*::*real*
⟨*proof*⟩

lemma *q-defer-nonneg*[*simp*]: $\$q\text{-}\{f|t\&x\} \geq 0$ **for** *f t* :: *real*
⟨*proof*⟩

lemma *q-defer-q*: $\$q\text{-}\{f|t\&x\} = \$q\text{-}\{f+t\ \&\ x\} - \$q\text{-}\{f\&x\}$ **if** $t \geq 0$ **for** *f t* :: *real*
⟨*proof*⟩

corollary *q-defer-le-1*[*simp*]: $\$q\text{-}\{f|t\&x\} \leq 1$ **if** $t \geq 0$ **for** *f t* :: *real*
⟨*proof*⟩

lemma *q-defer-PTx*: $\$q\text{-}\{f|t\&x\} = \mathcal{P}(\xi\ in\ \mathfrak{M}.\ f < T\ x\ \xi \wedge T\ x\ \xi \leq f + t \mid T\ x\ \xi > 0)$
if $t \geq 0$ **for** *f t* :: *real*
⟨*proof*⟩

lemma *q-defer-PX*: $\$q\text{-}\{f|t\&x\} = \mathcal{P}(\xi\ in\ \mathfrak{M}.\ x + f < X\ \xi \wedge X\ \xi \leq x + f + t)\ /\ \mathcal{P}(\xi\ in\ \mathfrak{M}.\ X\ \xi > x)$
if $f \geq 0$ $t \geq 0$ **for** *f t* :: *real*
⟨*proof*⟩

lemma *q-defer-old-0*: $\$q\text{-}\{f|t\&x\} = 0$ **if** $x+f \geq \$\psi$ $t \geq 0$ **for** *f t* :: *real*
⟨*proof*⟩

end

5.1.10 Properties of Cumulative Distributive Function for *X*

lemma *cdfX-continuous-unborn*[*simp*]: *continuous-on* {..*0*} (*cdf* (*distr* 𝔐 *borel X*))
⟨*proof*⟩

lemma *cdfX-differentiable-unborn*[*simp*]: (*cdf* (*distr* 𝔐 *borel X*)) *differentiable-on* {..*0*}
⟨*proof*⟩

lemma *cdfX-has-real-derivative-0-unborn*:
(*cdf* (*distr* 𝔐 *borel X*) *has-real-derivative 0*) (*at x*) **if** $x < 0$ **for** *x*::*real*
⟨*proof*⟩

lemma *cdfX-integrable-Icc*:
  $\textit{set-integrable lborel}\ \{a..b\}\ (\textit{cdf}\ (\textit{distr}\ \mathfrak{M}\ \textit{borel}\ X))$ **for** $a\ b :: \textit{real}$
  ⟨*proof*⟩

corollary *cdfX-integrable-on-Icc*:
  $\textit{cdf}\ (\textit{distr}\ \mathfrak{M}\ \textit{borel}\ X)\ \textit{integrable-on}\ \{a..b\}$ **for** $a\ b :: \textit{real}$
  ⟨*proof*⟩

lemma *cdfX-q*: $\textit{cdf}\ (\textit{distr}\ \mathfrak{M}\ \textit{borel}\ X)\ x = \$q\text{-}\{x\&0\}$ **if** $x \geq 0$ **for** $x{::}\textit{real}$
  ⟨*proof*⟩

5.1.11 Relations between $\$p\text{-}\{t\&x\}$ and $\$q\text{-}\{t\&x\}$

context
  fixes $x{::}\textit{real}$
  assumes *x-lt-psi*[*simp*]: $x < \$\psi$
begin

interpretation *alivex-PS*: $\textit{prob-space}\ \mathfrak{M} \downharpoonleft \textit{alive}\ x$
  ⟨*proof*⟩

interpretation *distrTx-RD*: $\textit{real-distribution distr}\ (\mathfrak{M} \downharpoonleft \textit{alive}\ x)\ \textit{borel}\ (T\ x)$ ⟨*proof*⟩

lemma *p-q-1*: $\$p\text{-}\{t\&x\} + \$q\text{-}\{t\&x\} = 1$ **for** $t{::}\textit{real}$
  ⟨*proof*⟩

lemma *q-defer-p*: $\$q\text{-}\{f|t\&x\} = \$p\text{-}\{f\&x\} - \$p\text{-}\{f{+}t\ \&\ x\}$ **if** $t \geq 0$ **for** $f\ t :: \textit{real}$
  ⟨*proof*⟩

lemma *q-defer-p-q-defer*: $\$p\text{-}\{f\&x\} * \$q\text{-}\{f'|t\ \&\ x{+}f\} = \$q\text{-}\{f{+}f'|t\ \&\ x\}$
  if $x{+}f < \$\psi\ f \geq 0\ f' \geq 0\ t \geq 0$ **for** $f\ f'\ t :: \textit{real}$
⟨*proof*⟩

lemma *q-defer-pq*: $\$q\text{-}\{f|t\&x\} = \$p\text{-}\{f\&x\} * \$q\text{-}\{t\ \&\ x{+}f\}$
  if $x{+}f < \$\psi\ t \geq 0\ f \geq 0$ **for** $f\ t :: \textit{real}$
  ⟨*proof*⟩

5.1.12 Properties of Life Expectation

lemma *e-nonneg*: $\$e`\circ\text{-}x \geq 0$
  ⟨*proof*⟩

lemma *e-P*: $\$e`\circ\text{-}x =$
  $\textit{MM-PS.expectation}\ (\lambda\xi.\ \textit{indicator}\ (\textit{alive}\ x)\ \xi * T\ x\ \xi)\ /\ \mathcal{P}(\xi\ \textit{in}\ \mathfrak{M}.\ T\ x\ \xi > 0)$
  ⟨*proof*⟩

proposition *nn-integral-T-p*:
  $(\int^{+}\xi.\ \textit{ennreal}\ (T\ x\ \xi)\ \partial(\mathfrak{M} \downharpoonleft \textit{alive}\ x)) = (\int^{+}t{\in}\{0..\}.\ \textit{ennreal}\ (\$p\text{-}\{t\&x\})\ \partial \textit{lborel})$
  ⟨*proof*⟩

lemma *nn-integral-T-pos*: $(\int^+\xi.\ ennreal\ (T\ x\ \xi)\ \partial(\mathfrak{M} \downharpoonright alive\ x)) > 0$
⟨*proof*⟩

lemma *e-pos-Tx*: $e‘∘-x > 0 **if** *integrable* $(\mathfrak{M} \downharpoonright alive\ x)\ (T\ x)$
⟨*proof*⟩

proposition *e-LBINT-p*: $e‘∘-x = (*LBINT t*:{*0*..}. $p-{*t*&*x*})
— Note that *0* = *0* holds when the integral diverges.
⟨*proof*⟩

corollary *e-integral-p*: $e‘∘-x = *integral* {*0*..} (λt. $p-{*t*&*x*})
— Note that *0* = *0* holds when the integral diverges.
⟨*proof*⟩

lemma *e-pos*: $e‘∘-x > *0* **if** *set-integrable lborel* {*0*..} (λt. $p-{*t*&*x*})
⟨*proof*⟩

corollary *e-pos'*: $e‘∘-x > *0* **if** ($\lambda t$. $p-{*t*&*x*}) *integrable-on* {*0*..}
⟨*proof*⟩

lemma *e-LBINT-p-Icc*: $e‘∘-x = (*LBINT t*:{*0*..*n*}. $p-{*t*&*x*}) **if** *x*+*n* ≥ $$\psi$ **for** *n*::*real*
⟨*proof*⟩

lemma *e-integral-p-Icc*: $e‘∘-x = *integral* {*0*..*n*} (λt. $p-{*t*&*x*}) **if** *x*+*n* ≥ $$\psi$ **for** *n*::*real*
⟨*proof*⟩

lemma *temp-e-le-n*: $e‘∘-{*x*:*n*} ≤ *n* **if** *n* ≥ *0* **for** *n*::*real*
⟨*proof*⟩

lemma *temp-e-P*: $e‘∘-{*x*:*n*} =
MM-PS.expectation ($\lambda\xi$. *indicator* (*alive x*) ξ * *min* (*T x* ξ) *n*) / $\mathcal{P}(\xi\ in\ \mathfrak{M}.\ T\ x\ \xi > 0)$
if *n* ≥ *0* **for** *n*::*real*
⟨*proof*⟩

lemma *temp-e-LBINT-p*: $e‘∘-{*x*:*n*} = (*LBINT t*:{*0*..*n*}. $p-{*t*&*x*}) **if** *n* ≥ *0* **for** *n*::*real*
⟨*proof*⟩

lemma *temp-e-integral-p*: $e‘∘-{*x*:*n*} = *integral* {*0*..*n*} (λt. $p-{*t*&*x*}) **if** *n* ≥ *0* **for** *n*::*real*
⟨*proof*⟩

lemma *e-eq-temp*: $e‘∘-x = $e‘∘-{*x*:*n*} **if** *n* ≥ *0* *x*+*n* ≥ $$\psi$ **for** *n*::*real*
⟨*proof*⟩

lemma *curt-e-P*: $e-*x* =

MM-PS.expectation ($\lambda\xi$. *indicator* (*alive x*) $\xi * \lfloor T\ x\ \xi \rfloor$) / $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. $T\ x\ \xi > 0)$
⟨*proof*⟩

lemma *curt-e-sum-P*: $e-x = ($\sum k$. $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. $T\ x\ \xi \geq k + 1 \mid T\ x\ \xi > 0))$
if *summable* (λk. $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. $T\ x\ \xi \geq k + 1 \mid T\ x\ \xi > 0))$
⟨*proof*⟩

lemma *curt-e-sum-P-finite*: $e-x = ($\sum k{<}n$. $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. $T\ x\ \xi \geq k + 1 \mid T\ x\ \xi > 0))$
if $x{+}n{+}1 > \$\psi$ **for** *n::nat*
⟨*proof*⟩

lemma *curt-e-sum-p*: $e-x = ($\sum k$. $p-{k+1&x})
if *summable* (λk. $p-{k+1&x}) $\bigwedge$*k::nat. isCont* (λt. $p-{t&x}) (*k+1*)
⟨*proof*⟩

lemma *curt-e-rec*: $e-x = $p-x * (1 + $e-(x+1))
if *summable* (λk. $p-{k+1&x}) $\bigwedge$*k::nat. isCont* (λt. $p-{t&x}) (*real k* + 1) *x+1* < $\$\psi$
⟨*proof*⟩

lemma *curt-e-sum-p-finite*: $e-x = ($\sum k{<}n$. $p-{k+1&x})
if $\bigwedge$*k::nat*. $k < n \Longrightarrow$ *isCont* (λt. $p-{t&x}) (*real k* + 1) $x{+}n{+}1 > \$\psi$ **for** *n::nat*
⟨*proof*⟩

lemma *temp-curt-e-P*: $e-{x:n} =
MM-PS.expectation ($\lambda\xi$. *indicator* (*alive x*) $\xi * \lfloor$*min* ($T\ x\ \xi$) $n\rfloor$) / $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. $T\ x\ \xi > 0)$
if $n \geq 0$ **for** *n::real*
⟨*proof*⟩

lemma *temp-curt-e-sum-P*: $e-{x:n} = ($\sum k{<}n$. $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. $T\ x\ \xi \geq k + 1 \mid T\ x\ \xi > 0))$ **for** *n::nat*
⟨*proof*⟩

corollary *curt-e-eq-temp*: $e-x = $e-{x:n} **if** $x{+}n{+}1 > \$\psi$ **for** *n::nat*
⟨*proof*⟩

lemma *temp-curt-e-sum-p*: $e-{x:n} = ($\sum k{<}n$. $p-{k+1&x})
if $\bigwedge$*k::nat*. $k < n \Longrightarrow$ *isCont* (λt. $p-{t&x}) (*real k* + 1) **for** *n::nat*
⟨*proof*⟩

lemma *temp-curt-e-rec*: $e-{x:n} = $p-x * (1 + $e-{x+1:n−1})
if $\bigwedge$*k::nat*. $k < n \Longrightarrow$ *isCont* (λt. $p-{t&x}) (*real k* + 1) $x{+}1 < \$\psi$ $n \neq 0$ **for** *n::nat*
⟨*proof*⟩

end

lemma *p-set-integrable-shift*:
set-integrable lborel {*0..*} ($\lambda t.$ \$*p*-{*t*&*0*}) $\longleftrightarrow$ *set-integrable lborel* {*0..*} ($\lambda t.$ \$*p*-{*t*&*x*})
if $x <$ \$$\psi$ **for** *x*::*real*
⟨*proof*⟩

lemma *e-p-e*: \$*e*'∘-*x* = \$*e*'∘-{*x*:*n*} + \$*p*-{*n*&*x*} ∗ \$*e*'∘-(*x*+*n*)
if *set-integrable lborel* {*0..*} ($\lambda t.$ \$*p*-{*t*&*x*}) $n \geq 0$ $x+n <$ \$$\psi$ **for** *x n* :: *real*
⟨*proof*⟩

proposition *x-ex-mono*: *x* + \$*e*'∘-*x* ≤ *y* + \$*e*'∘-*y* **if** $x \leq y$ $y <$ \$$\psi$ **for** *x y* :: *real*
⟨*proof*⟩

proposition *x-ex-const-equiv*: *x* + \$*e*'∘-*x* = *y* + \$*e*'∘-*y* ⟷ \$*q*-{*y*−*x*&*x*} = *0*
if *set-integrable lborel* {*0..*} ($\lambda t.$ \$*p*-{*t*&*0*}) $x \leq y$ $y <$ \$$\psi$ **for** *x y* :: *real*
⟨*proof*⟩

end

5.2 Piecewise Differentiable Survival Function

locale *smooth-survival-function* = *survival-model* +
assumes *ccdfX-piecewise-differentiable*[*simp*]:
(*ccdf* (*distr* 𝔐 *borel X*)) *piecewise-differentiable-on UNIV*
begin

interpretation *distrX-RD*: *real-distribution distr* 𝔐 *borel X*
⟨*proof*⟩

5.2.1 Properties of Survival Function for *X*

lemma *ccdfX-continuous*[*simp*]: *continuous-on UNIV* (*ccdf* (*distr* 𝔐 *borel X*))
⟨*proof*⟩

corollary *ccdfX-borel-measurable*[*measurable*]: *ccdf* (*distr* 𝔐 *borel X*) ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *ccdfX-nondifferentiable-finite-set*[*simp*]:
finite {*x*. ¬ *ccdf* (*distr* 𝔐 *borel X*) *differentiable at x*}
⟨*proof*⟩

lemma *ccdfX-differentiable-open-set*: *open* {*x*. *ccdf* (*distr* 𝔐 *borel X*) *differentiable at x*}
⟨*proof*⟩

lemma *ccdfX-differentiable-borel-set*[*measurable*, *simp*]:
{*x*. *ccdf* (*distr* 𝔐 *borel X*) *differentiable at x*} ∈ *sets borel*

⟨*proof*⟩

lemma *ccdfX-differentiable-AE*:
AE x in lborel. (*ccdf* (*distr* 𝔐 *borel X*)) *differentiable at x*
⟨*proof*⟩

lemma *deriv-ccdfX-measurable*[*measurable*]: *deriv* (*ccdf* (*distr* 𝔐 *borel X*)) ∈ *borel-measurable borel*
⟨*proof*⟩

5.2.2 Properties of Cumulative Distributive Function for *X*

lemma *cdfX-piecewise-differentiable*[*simp*]:
(*cdf* (*distr* 𝔐 *borel X*)) *piecewise-differentiable-on UNIV*
⟨*proof*⟩

lemma *cdfX-continuous*[*simp*]: *continuous-on UNIV* (*cdf* (*distr* 𝔐 *borel X*))
⟨*proof*⟩

corollary *cdfX-borel-measurable*[*measurable*]: *cdf* (*distr* 𝔐 *borel X*) ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *cdfX-nondifferentiable-finite-set*[*simp*]:
finite {*x*. ¬ *cdf* (*distr* 𝔐 *borel X*) *differentiable at x*}
⟨*proof*⟩

lemma *cdfX-differentiable-open-set*: *open* {*x*. *cdf* (*distr* 𝔐 *borel X*) *differentiable at x*}
⟨*proof*⟩

lemma *cdfX-differentiable-borel-set*[*measurable*, *simp*]:
{*x*. *cdf* (*distr* 𝔐 *borel X*) *differentiable at x*} ∈ *sets borel*
⟨*proof*⟩

lemma *cdfX-differentiable-AE*:
AE x in lborel. (*cdf* (*distr* 𝔐 *borel X*)) *differentiable at x*
⟨*proof*⟩

lemma *deriv-cdfX-measurable*[*measurable*]: *deriv* (*cdf* (*distr* 𝔐 *borel X*)) ∈ *borel-measurable borel*
⟨*proof*⟩

5.2.3 Introduction of Probability Density Functions of *X* and *T*(*x*)

definition *pdfX* :: *real* ⇒ *real*
where *pdfX x* ≡ *if cdf* (*distr* 𝔐 *borel X*) *differentiable at x*
then deriv (*cdf* (*distr* 𝔐 *borel X*)) *x else 0*
— This function is defined to be always nonnegative for future application.

definition *pdfT* :: *real* ⇒ *real* ⇒ *real*
  where *pdfT x t* ≡ *if cdf* (*distr* (𝔐 ⇂ *alive x*) *borel* (*T x*)) *differentiable at t*
  then deriv (*cdf* (*distr* (𝔐 ⇂ *alive x*) *borel* (*T x*))) *t else 0*
  — This function is defined to be always nonnegative for future application.

lemma *pdfX-measurable*[*measurable*]: *pdfX* ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *distributed-pdfX*: *distributed* 𝔐 *lborel X pdfX*
⟨*proof*⟩

lemma *pdfT0-X*: *pdfT 0* = *pdfX*
⟨*proof*⟩

5.2.4 Properties of Survival Function for $T(x)$

context
  fixes *x*::*real*
  assumes *x-lt-psi*[*simp*]: *x* < $ψ
begin

interpretation *alivex-PS*: *prob-space* 𝔐 ⇂ *alive x*
⟨*proof*⟩

interpretation *distrTx-RD*: *real-distribution distr* (𝔐 ⇂ *alive x*) *borel* (*T x*) ⟨*proof*⟩

lemma *ccdfTx-continuous-on-nonneg*[*simp*]:
  continuous-on {*0*..} (*ccdf* (*distr* (𝔐 ⇂ *alive x*) *borel* (*T x*)))
⟨*proof*⟩

lemma *ccdfTx-continuous*[*simp*]: *continuous-on UNIV* (*ccdf* (*distr* (𝔐 ⇂ *alive x*) *borel* (*T x*)))
⟨*proof*⟩

corollary *ccdfTx-borel-measurable*[*measurable*]:
  ccdf (*distr* (𝔐 ⇂ *alive x*) *borel* (*T x*)) ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *ccdfTx-nondifferentiable-finite-set*[*simp*]:
  finite {*t*. ¬ *ccdf* (*distr* (𝔐 ⇂ *alive x*) *borel* (*T x*)) *differentiable at t*}
⟨*proof*⟩

lemma *ccdfTx-differentiable-open-set*:
  open {*t*. *ccdf* (*distr* (𝔐 ⇂ *alive x*) *borel* (*T x*)) *differentiable at t*}
⟨*proof*⟩

lemma *ccdfTx-differentiable-borel-set*[*measurable*, *simp*]:
  {*t*. *ccdf* (*distr* (𝔐 ⇂ *alive x*) *borel* (*T x*)) *differentiable at t*} ∈ *sets borel*

⟨*proof*⟩

lemma *ccdfTx-differentiable-AE*:
AE t in lborel. (*ccdf* (*distr* ($\mathfrak{M}$ ⇂ *alive x*) *borel* (*T x*))) *differentiable at t*
⟨*proof*⟩

lemma *ccdfTx-piecewise-differentiable*[*simp*]:
(*ccdf* (*distr* ($\mathfrak{M}$ ⇂ *alive x*) *borel* (*T x*))) *piecewise-differentiable-on UNIV*
⟨*proof*⟩

lemma *deriv-ccdfTx-measurable*[*measurable*]:
deriv (*ccdf* (*distr* ($\mathfrak{M}$ ⇂ *alive x*) *borel* (*T x*))) ∈ *borel-measurable borel*
⟨*proof*⟩

5.2.5 Properties of Cumulative Distributive Function for $T(x)$

lemma *cdfTx-continuous*[*simp*]:
continuous-on UNIV (*cdf* (*distr* ($\mathfrak{M}$ ⇂ *alive x*) *borel* (*T x*)))
⟨*proof*⟩

corollary *cdfTx-borel-measurable*[*measurable*]:
cdf (*distr* ($\mathfrak{M}$ ⇂ *alive x*) *borel* (*T x*)) ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *cdfTx-nondifferentiable-finite-set*[*simp*]:
finite {*t*. ¬ *cdf* (*distr* ($\mathfrak{M}$ ⇂ *alive x*) *borel* (*T x*)) *differentiable at t*}
⟨*proof*⟩

lemma *cdfTx-differentiable-open-set*:
open {*t*. *cdf* (*distr* ($\mathfrak{M}$ ⇂ *alive x*) *borel* (*T x*)) *differentiable at t*}
⟨*proof*⟩

lemma *cdfTx-differentiable-borel-set*[*measurable*, *simp*]:
{*t*. *cdf* (*distr* ($\mathfrak{M}$ ⇂ *alive x*) *borel* (*T x*)) *differentiable at t*} ∈ *sets borel*
⟨*proof*⟩

lemma *cdfTx-differentiable-AE*:
AE t in lborel. (*cdf* (*distr* ($\mathfrak{M}$ ⇂ *alive x*) *borel* (*T x*))) *differentiable at t*
⟨*proof*⟩

lemma *cdfTx-piecewise-differentiable*[*simp*]:
(*cdf* (*distr* ($\mathfrak{M}$ ⇂ *alive x*) *borel* (*T x*))) *piecewise-differentiable-on UNIV*
⟨*proof*⟩

lemma *deriv-cdfTx-measurable*[*measurable*]:
deriv (*cdf* (*distr* ($\mathfrak{M}$ ⇂ *alive x*) *borel* (*T x*))) ∈ *borel-measurable borel*
⟨*proof*⟩

5.2.6 Properties of Probability Density Function of $T(x)$

lemma *pdfTx-nonneg*: *pdfT x t* $\geq$ *0* **for** *t::real*
⟨*proof*⟩

lemma *pdfTx-neg-0*: *pdfT x t* = *0* **if** *t* < *0* **for** *t::real*
⟨*proof*⟩

lemma *pdfTx-0-0*: *pdfT x 0* = *0*
⟨*proof*⟩

lemma *pdfTx-nonpos-0*: *pdfT x t* = *0* **if** *t* $\leq$ *0* **for** *t::real*
⟨*proof*⟩

lemma *pdfTx-beyond-0*: *pdfT x t* = *0* **if** *x+t* $\geq$ \$$\psi$ **for** *t::real*
⟨*proof*⟩

lemma *pdfTx-pdfX*: *pdfT x t* = *pdfX* (*x+t*) / $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *X* ξ > *x*) **if** *t* > *0* **for** *t::real*
⟨*proof*⟩

lemma *pdfTx-measurable*[*measurable*]: *pdfT x* $\in$ *borel-measurable borel*
⟨*proof*⟩

lemma *distributed-pdfTx*: *distributed* ($\mathfrak{M}$ $\downharpoonright$ *alive x*) *lborel* (*T x*) (*pdfT x*)
⟨*proof*⟩

lemma *nn-integral-pdfTx-1*: ($\int^{+}$*s. pdfT x s* ∂*lborel*) = *1*
⟨*proof*⟩

corollary *has-bochner-integral-pdfTx-1*: *has-bochner-integral lborel* (*pdfT x*) *1*
⟨*proof*⟩

corollary *LBINT-pdfTx-1*: (*LBINT s. pdfT x s*) = *1*
⟨*proof*⟩

corollary *pdfTx-has-integral-1*: (*pdfT x has-integral 1*) *UNIV*
⟨*proof*⟩

lemma *set-nn-integral-pdfTx-1*: ($\int^{+}$*s*$\in${*0..*}. *pdfT x s* ∂*lborel*) = *1*
⟨*proof*⟩

corollary *has-bochner-integral-pdfTx-1-nonpos*:
has-bochner-integral lborel (λ*s. pdfT x s* $*$ *indicator* {*0..*} *s*) *1*
⟨*proof*⟩

corollary *set-LBINT-pdfTx-1*: (*LBINT s*:{*0..*}. *pdfT x s*) = *1*
⟨*proof*⟩

corollary *pdfTx-has-integral-1-nonpos*: (*pdfT x has-integral 1*) {*0..*}

⟨*proof*⟩

lemma *set-nn-integral-pdfTx-PTx*: $(\int^+ s{\in}A.\ pdfT\ x\ s\ \partial lborel) = \mathcal{P}(\xi\ in\ \mathfrak{M}.\ T\ x\ \xi \in A \mid T\ x\ \xi > 0)$
if $A \in sets\ lborel$ **for** $A :: real\ set$
⟨*proof*⟩

lemma *pdfTx-set-integrable*: *set-integrable lborel A (pdfT x)* **if** $A \in sets\ lborel$
⟨*proof*⟩

lemma *set-integral-pdfTx-PTx*: $(LBINT\ s{:}A.\ pdfT\ x\ s) = \mathcal{P}(\xi\ in\ \mathfrak{M}.\ T\ x\ \xi \in A \mid T\ x\ \xi > 0)$
if $A \in sets\ lborel$ **for** $A :: real\ set$
⟨*proof*⟩

lemma *pdfTx-has-integral-PTx*: $(pdfT\ x\ has\text{-}integral\ \mathcal{P}(\xi\ in\ \mathfrak{M}.\ T\ x\ \xi \in A \mid T\ x\ \xi > 0))\ A$
if $A \in sets\ lborel$ **for** $A :: real\ set$
⟨*proof*⟩

corollary *pdfTx-has-integral-PTx-Icc*:
$(pdfT\ x\ has\text{-}integral\ \mathcal{P}(\xi\ in\ \mathfrak{M}.\ a \le T\ x\ \xi \wedge T\ x\ \xi \le b \mid T\ x\ \xi > 0))\ \{a..b\}$ **for** $a\ b :: real$
⟨*proof*⟩

corollary *pdfTx-integrable-on-Icc*: *pdfT x integrable-on* $\{a..b\}$ **for** $a\ b :: real$
⟨*proof*⟩

end

5.2.7 Properties of Probability Density Function of X

lemma *pdfX-nonneg*: $pdfX\ x \ge 0$ **for** $x{::}real$
⟨*proof*⟩

lemma *pdfX-nonpos-0*: $pdfX\ x = 0$ **if** $x \le 0$ **for** $x{::}real$
⟨*proof*⟩

lemma *pdfX-beyond-0*: $pdfX\ x = 0$ **if** $x \ge \$\psi$ **for** $x{::}real$
⟨*proof*⟩

lemma *nn-integral-pdfX-1*: $integral^N\ lborel\ pdfX = 1$
⟨*proof*⟩

corollary *has-bochner-integral-pdfX-1*: *has-bochner-integral lborel pdfX 1*
⟨*proof*⟩

corollary *LBINT-pdfX-1*: $(LBINT\ s.\ pdfX\ s) = 1$
⟨*proof*⟩

corollary *pdfX-has-integral-1*: (*pdfX has-integral 1*) *UNIV*
⟨*proof*⟩

lemma *set-nn-integral-pdfX-PX*: *set-nn-integral lborel A pdfX* = $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *X* ξ ∈ *A*)
if *A* ∈ *sets lborel* **for** *A* :: *real set*
⟨*proof*⟩

lemma *pdfX-set-integrable*: *set-integrable lborel A pdfX* **if** *A* ∈ *sets lborel* **for** *A* :: *real set*
⟨*proof*⟩

lemma *set-integral-pdfX-PX*: (*LBINT s*:*A*. *pdfX s*) = $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *X* ξ ∈ *A*)
if *A* ∈ *sets lborel* **for** *A* :: *real set*
⟨*proof*⟩

lemma *pdfX-has-integral-PX*: (*pdfX has-integral* $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *X* ξ ∈ *A*)) *A*
if *A* ∈ *sets lborel* **for** *A* :: *real set*
⟨*proof*⟩

corollary *pdfX-has-integral-PX-Icc*: (*pdfX has-integral* $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *a* ≤ *X* ξ ∧ *X* ξ ≤ *b*)) {*a*..*b*}
for *a b* :: *real*
⟨*proof*⟩

corollary *pdfX-integrable-on-Icc*: *pdfX integrable-on* {*a*..*b*} **for** *a b* :: *real*
⟨*proof*⟩

5.2.8 Relations between Life Expectation and Probability Density Function

context
fixes *x*::*real*
assumes *x-lt-psi*[*simp*]: *x* < \$$\psi$
begin

interpretation *alivex-PS*: *prob-space* $\mathfrak{M}$ ↓ *alive x*
⟨*proof*⟩

interpretation *distrTx-RD*: *real-distribution distr* ($\mathfrak{M}$ ↓ *alive x*) *borel* (*T x*) ⟨*proof*⟩

proposition *nn-integral-T-pdfT*:
($\int^+\xi$. *ennreal* (*g* (*T x* ξ)) ∂($\mathfrak{M}$ ↓ *alive x*)) = ($\int^+ s$∈{*0*..}. *ennreal* (*pdfT x s* ∗ *g s*) ∂*lborel*)
if *g* ∈ *borel-measurable lborel* **for** *g* :: *real* ⇒ *real*
⟨*proof*⟩

lemma *expectation-LBINT-pdfT-nonneg*:

alivex-PS.expectation ($\lambda\xi$. *g* (*T x* ξ)) = (*LBINT s*:{*0*..}. *pdfT x s* ∗ *g s*)
if $\bigwedge$*s*. *s* ≥ *0* ⟹ *g s* ≥ *0 g* ∈ *borel-measurable lborel* **for** *g* :: *real* ⇒ *real*
— Note that *0* = *0* holds when the integral diverges.
⟨*proof*⟩

corollary *expectation-integral-pdfT-nonneg*:
alivex-PS.expectation ($\lambda\xi$. *g* (*T x* ξ)) = *integral* {*0*..} (λ*s*. *pdfT x s* ∗ *g s*)
if $\bigwedge$*s*. *s* ≥ *0* ⟹ *g s* ≥ *0 g* ∈ *borel-measurable lborel* **for** *g* :: *real* ⇒ *real*
— Note that *0* = *0* holds when the integral diverges.
⟨*proof*⟩

proposition *expectation-LBINT-pdfT*:
alivex-PS.expectation ($\lambda\xi$. *g* (*T x* ξ)) = (*LBINT s*:{*0*..}. *pdfT x s* ∗ *g s*)
if *set-integrable lborel* {*0*..} (λ*s*. *pdfT x s* ∗ *g s*) *g* ∈ *borel-measurable lborel*
for *g* :: *real* ⇒ *real*
⟨*proof*⟩

corollary *expectation-integral-pdfT*:
alivex-PS.expectation ($\lambda\xi$. *g* (*T x* ξ)) = *integral* {*0*..} (λ*s*. *pdfT x s* ∗ *g s*)
if (λ*s*. *pdfT x s* ∗ *g s*) *absolutely-integrable-on* {*0*..} *g* ∈ *borel-measurable lborel*
for *g* :: *real* ⇒ *real*
⟨*proof*⟩

corollary *e-LBINT-pdfT*: \$*e*'∘-*x* = (*LBINT s*:{*0*..}. *pdfT x s* ∗ *s*)
— Note that *0* = *0* holds when the life expectation diverges.
⟨*proof*⟩

corollary *e-integral-pdfT*: \$*e*'∘-*x* = *integral* {*0*..} (λ*s*. *pdfT x s* ∗ *s*)
— Note that *0* = *0* holds when the life expectation diverges.
⟨*proof*⟩

end

corollary *e-LBINT-pdfX*: \$*e*'∘-*0* = (*LBINT x*:{*0*..}. *pdfX x* ∗ *x*)
— Note that *0* = *0* holds when the life expectation diverges.
⟨*proof*⟩

corollary *e-integral-pdfX*: \$*e*'∘-*0* = *integral* {*0*..} (λ*x*. *pdfX x* ∗ *x*)
— Note that *0* = *0* holds when the life expectation diverges.
⟨*proof*⟩

5.2.9 Introduction of Force of Mortality

definition *force-mortal* :: *real* ⇒ *real* (\$$\mu$'-- [*101*] *200*)
where \$$\mu$-*x* ≡ *MM-PS.hazard-rate X x*

lemma *mu-pdfX*: \$$\mu$-*x* = *pdfX x* / *ccdf* (*distr* $\mathfrak{M}$ *borel X*) *x*
if (*cdf* (*distr* $\mathfrak{M}$ *borel X*)) *differentiable at x* **for** *x*::*real*
⟨*proof*⟩

lemma *mu-unborn-0*: $\$\mu\text{-}x = 0$ **if** $x < 0$ **for** *x::real*
⟨*proof*⟩

lemma *mu-beyond-0*: $\$\mu\text{-}x = 0$ **if** $x \geq \$\psi$ **for** *x::real*
— Note that division by *0* is defined as *0* in Isabelle/HOL.
⟨*proof*⟩

lemma *mu-nonneg-differentiable*: $\$\mu\text{-}x \geq 0$
if (*cdf* (*distr* $\mathfrak{M}$ *borel X*)) *differentiable at x* **for** *x::real*
⟨*proof*⟩

lemma *mu-nonneg-AE*: *AE x in lborel*. $\$\mu\text{-}x \geq 0$
⟨*proof*⟩

lemma *mu-measurable*[*measurable*]: $(\lambda x.\ \$\mu\text{-}x) \in$ *borel-measurable borel*
⟨*proof*⟩

lemma *mu-deriv-ccdf*: $\$\mu\text{-}x = -$ *deriv* (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) *x* / *ccdf* (*distr* $\mathfrak{M}$ *borel X*) *x*
if (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) *differentiable at x* $x < \$\psi$ **for** *x::real*
⟨*proof*⟩

lemma *mu-deriv-ln*: $\$\mu\text{-}x = -$ *deriv* (λx. *ln* (*ccdf* (*distr* $\mathfrak{M}$ *borel X*) *x*)) *x*
if (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) *differentiable at x* $x < \$\psi$ **for** *x::real*
⟨*proof*⟩

lemma *p-exp-integral-mu*: $\$p\text{-}\{t\&x\} = exp\ (-\ integral\ \{x..x+t\}\ (\lambda y.\ \$\mu\text{-}y))$
if $x \geq 0$ $t \geq 0$ $x+t < \$\psi$ **for** *x t :: real*
⟨*proof*⟩

corollary *ccdfX-exp-integral-mu*: *ccdf* (*distr* $\mathfrak{M}$ *borel X*) $x = exp\ (-\ integral\ \{0..x\}\ (\lambda y.\ \$\mu\text{-}y))$
if $0 \leq x \wedge x < \$\psi$ **for** *x::real*
⟨*proof*⟩

5.2.10 Properties of Force of Mortality

context
fixes *x::real*
assumes *x-lt-psi*[*simp*]: $x < \$\psi$
begin

interpretation *alivex-PS*: *prob-space* $\mathfrak{M} \downharpoonright$ *alive x*
⟨*proof*⟩

interpretation *distrTx-RD*: *real-distribution distr* ($\mathfrak{M} \downharpoonright$ *alive x*) *borel* (*T x*) ⟨*proof*⟩

lemma *hazard-rate-Tx-mu*: *alivex-PS.hazard-rate* (*T x*) $t = \$\mu\text{-}(x+t)$

if $t \geq 0$ $x+t < \$\psi$ **for** t::*real*
⟨*proof*⟩

lemma *pdfTx-p-mu*: $pdfT\ x\ t = \$p\text{-}\{t\&x\} * \$\mu\text{-}(x+t)$
if $(cdf\ (distr\ (\mathfrak{M} \downharpoonright alive\ x)\ borel\ (T\ x)))$ *differentiable at* t $t > 0$ **for** t::*real*
⟨*proof*⟩

lemma *deriv-t-p-mu*: $deriv\ (\lambda s.\ \$p\text{-}\{s\&x\})\ t = -\ \$p\text{-}\{t\&x\} * \$\mu\text{-}(x+t)$
if $(\lambda s.\ \$p\text{-}\{s\&x\})$ *differentiable at* t $t > 0$ **for** t::*real*
⟨*proof*⟩

lemma *pdfTx-p-mu-AE*: *AE s in lborel.* $s > 0 \longrightarrow pdfT\ x\ s = \$p\text{-}\{s\&x\} * \$\mu\text{-}(x+s)$
⟨*proof*⟩

lemma *LBINT-p-mu-q-defer*: $(LBINT\ s{:}\{f{<}..f+t\}.\ \$p\text{-}\{s\&x\} * \$\mu\text{-}(x+s)) = \$q\text{-}\{f|t\&x\}$
if $t \geq 0$ $f \geq 0$ **for** $t\ f$ :: *real*
⟨*proof*⟩

corollary *LBINT-p-mu-q*: $(LBINT\ s{:}\{0{<}..t\}.\ \$p\text{-}\{s\&x\} * \$\mu\text{-}(x+s)) = \$q\text{-}\{t\&x\}$
if $t \geq 0$ **for** t::*real*
⟨*proof*⟩

lemma *set-integrable-p-mu*: *set-integrable lborel* $\{f{<}..f+t\}$ $(\lambda s.\ \$p\text{-}\{s\&x\} * \$\mu\text{-}(x+s))$
if $t \geq 0$ $f \geq 0$ **for** $t\ f$ :: *real*
⟨*proof*⟩

lemma *p-mu-has-integral-q-defer-Ioc*:
$((\lambda s.\ \$p\text{-}\{s\&x\} * \$\mu\text{-}(x+s))$ *has-integral* $\$q\text{-}\{f|t\&x\})$ $\{f{<}..f+t\}$
if $t \geq 0$ $f \geq 0$ **for** $t\ f$ :: *real*
⟨*proof*⟩

lemma *p-mu-has-integral-q-defer-Icc*:
$((\lambda s.\ \$p\text{-}\{s\&x\} * \$\mu\text{-}(x+s))$ *has-integral* $\$q\text{-}\{f|t\&x\})$ $\{f..f+t\}$ **if** $t \geq 0$ $f \geq 0$ **for** $t\ f$ :: *real*
⟨*proof*⟩

corollary *p-mu-has-integral-q-Icc*:
$((\lambda s.\ \$p\text{-}\{s\&x\} * \$\mu\text{-}(x+s))$ *has-integral* $\$q\text{-}\{t\&x\})$ $\{0..t\}$ **if** $t \geq 0$ **for** t::*real*
⟨*proof*⟩

corollary *p-mu-integrable-on-Icc*:
$(\lambda s.\ \$p\text{-}\{s\&x\} * \$\mu\text{-}(x+s))$ *integrable-on* $\{0..t\}$ **if** $t \geq 0$ **for** t::*real*
⟨*proof*⟩

lemma *e-ennreal-p-mu*: $(\int^{+}\xi.\ ennreal\ (T\ x\ \xi)\ \partial(\mathfrak{M} \downharpoonright alive\ x)) =$
$(\int^{+}s{\in}\{0..\}.\ ennreal\ (\$p\text{-}\{s\&x\} * \$\mu\text{-}(x+s) * s)\ \partial lborel)$
⟨*proof*⟩

lemma *e-LBINT-p-mu*: $\$e\text{`}\circ\text{-}x = (LBINT\ s{:}\{0..\}.\ \$p\text{-}\{s\&x\} * \$\mu\text{-}(x+s) * s)$

— Note that *0* = *0* holds when the life expectation diverges.
⟨*proof*⟩

lemma *e-integral-p-mu*: $\$e`\circ\text{-}x = integral\ \{0..\}\ (\lambda s.\ \$p\text{-}\{s\&x\} * \$\mu\text{-}(x+s) * s)$
— Note that *0* = *0* holds when the life expectation diverges.
⟨*proof*⟩

end

lemma *p-has-real-derivative-x-ccdfX*:
$((\lambda y.\ \$p\text{-}\{t\&y\})\ has\text{-}real\text{-}derivative$
$((deriv\ svl\ (x+t) * svl\ x - svl\ (x+t) * deriv\ svl\ x) \ /\ (svl\ x)^2))\ (at\ x)$
if $svl \equiv ccdf\ (distr\ \mathfrak{M}\ borel\ X)$ *svl differentiable at x svl differentiable at* $(x+t)$
$t \geq 0$ $x < \$\psi$ **for** *x t* :: *real*
⟨*proof*⟩

lemma *p-has-real-derivative-x-p-mu*:
$((\lambda y.\ \$p\text{-}\{t\&y\})\ has\text{-}real\text{-}derivative\ \$p\text{-}\{t\&x\} * (\$\mu\text{-}x - \$\mu\text{-}(x+t)))\ (at\ x)$
if $ccdf\ (distr\ \mathfrak{M}\ borel\ X)$ *differentiable at x* $ccdf\ (distr\ \mathfrak{M}\ borel\ X)$ *differentiable at* $(x+t)$
$t \geq 0$ $x < \$\psi$ **for** *x t* :: *real*
⟨*proof*⟩

corollary *deriv-x-p-mu*: $deriv\ (\lambda y.\ \$p\text{-}\{t\&y\})\ x = \$p\text{-}\{t\&x\} * (\$\mu\text{-}x - \$\mu\text{-}(x+t))$
if $ccdf\ (distr\ \mathfrak{M}\ borel\ X)$ *differentiable at x* $ccdf\ (distr\ \mathfrak{M}\ borel\ X)$ *differentiable at* $(x+t)$
$t \geq 0$ $x < \$\psi$ **for** *x t* :: *real*
⟨*proof*⟩

lemma *e-has-derivative-mu-e*: $((\lambda x.\ \$e`\circ\text{-}x)\ has\text{-}real\text{-}derivative\ (\$\mu\text{-}x * \$e`\circ\text{-}x - 1))\ (at\ x)$
if $\bigwedge x.\ x \in \{a <..< b\} \Longrightarrow set\text{-}integrable\ lborel\ \{x..\}\ (ccdf\ (distr\ \mathfrak{M}\ borel\ X))$
$ccdf\ (distr\ \mathfrak{M}\ borel\ X)$ *differentiable at x* $x \in \{a <..< b\}$ $b \leq \$\psi$
for *a b x* :: *real*
⟨*proof*⟩

corollary *e-has-derivative-mu-e'*: $((\lambda x.\ \$e`\circ\text{-}x)\ has\text{-}real\text{-}derivative\ (\$\mu\text{-}x * \$e`\circ\text{-}x - 1))\ (at\ x)$
if $\bigwedge x.\ x \in \{a <..< b\} \Longrightarrow ccdf\ (distr\ \mathfrak{M}\ borel\ X)\ integrable\text{-}on\ \{x..\}$
$ccdf\ (distr\ \mathfrak{M}\ borel\ X)$ *differentiable at x* $x \in \{a <..< b\}$ $b \leq \$\psi$
for *a b x* :: *real*
⟨*proof*⟩

5.2.11 Properties of Curtate Life Expectation

context
fixes *x*::*real*
assumes *x-lt-psi*[*simp*]: $x < \$\psi$
begin

lemma *isCont-p-nat*: *isCont* ($\lambda t.$ \$*p*-{*t*&*x*}) (*k* + (*1*::*real*)) **for** *k*::*nat*
⟨*proof*⟩

lemma *curt-e-sum-p-smooth*: \$*e*-*x* = ($\sum$ *k*. \$*p*-{*k*+*1*&*x*}) **if** *summable* ($\lambda k.$ \$*p*-{*k*+*1*&*x*})
⟨*proof*⟩

lemma *curt-e-rec-smooth*: \$*e*-*x* = \$*p*-*x* ∗ (*1* + \$*e*-(*x*+*1*)) **if** *summable* ($\lambda k.$ \$*p*-{*k*+*1*&*x*}) *x*+*1* < \$$\psi$
⟨*proof*⟩

lemma *curt-e-sum-p-finite-smooth*: \$*e*-*x* = ($\sum$ *k*<*n*. \$*p*-{*k*+*1*&*x*}) **if** *x*+*n*+*1* > \$$\psi$ **for** *n*::*nat*
⟨*proof*⟩

lemma *temp-curt-e-sum-p-smooth*: \$*e*-{*x*:*n*} = ($\sum$ *k*<*n*. \$*p*-{*k*+*1*&*x*}) **for** *n*::*nat*
⟨*proof*⟩

lemma *temp-curt-e-rec-smooth*: \$*e*-{*x*:*n*} = \$*p*-*x* ∗ (*1* + \$*e*-{*x*+*1*:*n*−*1*})
if *x*+*1* < \$$\psi$ *n* ≠ *0* **for** *n*::*nat*
⟨*proof*⟩

end

end

5.3 Limited Survival Function

locale *limited-survival-function* = *survival-model* +
assumes *psi-limited*[*simp*]: \$$\psi$ < ∞
begin

definition *ult-age* :: *nat* (\$$\omega$)
where \$$\omega$ ≡ *LEAST x*::*nat*. *ccdf* (*distr* $\mathfrak{M}$ *borel X*) *x* = *0*
— the conventional notation for ultimate age

lemma *ccdfX-ceil-psi-0*: *ccdf* (*distr* $\mathfrak{M}$ *borel X*) ⌈*real-of-ereal* \$$\psi$⌉ = *0*
⟨*proof*⟩

lemma *ccdfX-omega-0*: *ccdf* (*distr* $\mathfrak{M}$ *borel X*) \$$\omega$ = *0*
⟨*proof*⟩

corollary *psi-le-omega*: \$$\psi$ ≤ \$$\omega$
⟨*proof*⟩

corollary *omega-pos*: \$$\omega$ > *0*
⟨*proof*⟩

lemma *omega-ceil-psi*: \$$\omega$ = ⌈*real-of-ereal* \$$\psi$⌉

$\langle proof \rangle$

lemma *ccdfX-0-equiv-nat*: *ccdf* (*distr* $\mathfrak{M}$ *borel X*) *x* = *0* $\longleftrightarrow$ *x* $\geq$ \$$\omega$ **for** *x*::*nat*
$\langle proof \rangle$

lemma *psi-le-iff-omega-le*: \$$\psi$ $\leq$ *x* $\longleftrightarrow$ \$$\omega$ $\leq$ *x* **for** *x*::*nat*
$\langle proof \rangle$

context
fixes *x*::*nat*
assumes *x-lt-omega*[*simp*]: *x* < \$$\omega$
begin

lemma *x-lt-psi*[*simp*]: *x* < \$$\psi$
$\langle proof \rangle$

lemma *p-0-1-nat*: \$*p*-{*0*&*x*} = *1*
$\langle proof \rangle$

lemma *p-0-equiv-nat*: \$*p*-{*t*&*x*} = *0* $\longleftrightarrow$ *x*+*t* $\geq$ \$$\omega$ **for** *t*::*nat*
$\langle proof \rangle$

lemma *q-0-0-nat*: \$*q*-{*0*&*x*} = *0*
$\langle proof \rangle$

lemma *q-1-equiv-nat*: \$*q*-{*t*&*x*} = *1* $\longleftrightarrow$ *x*+*t* $\geq$ \$$\omega$ **for** *t*::*nat*
$\langle proof \rangle$

lemma *q-defer-old-0-nat*: \$*q*-{*f*|*t*&*x*} = *0* **if** \$$\omega$ $\leq$ *x*+*f* **for** *f t* :: *nat*
$\langle proof \rangle$

lemma *curt-e-sum-P-finite-nat*: \$*e*-*x* = ($\sum$ *k*<*n*. $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *T x* ξ $\geq$ *k* + *1* | *T x* ξ > *0*))
if *x*+*n* $\geq$ \$$\omega$ **for** *n*::*nat*
$\langle proof \rangle$

lemma *curt-e-sum-p-finite-nat*: \$*e*-*x* = ($\sum$ *k*<*n*. \$*p*-{*k*+*1*&*x*})
if $\bigwedge$*k*::*nat*. *k* < *n* $\Longrightarrow$ *isCont* (λ*t*. \$*p*-{*t*&*x*}) (*real k* + *1*) *x*+*n* $\geq$ \$$\omega$ **for** *n*::*nat*
$\langle proof \rangle$

end

lemma *q-omega-1*: \$*q*-(\$$\omega$−*1*) = *1*
$\langle proof \rangle$

end

end
theory *Life-Table*

imports *Survival-Model*
begin

6 Life Table

Define a life table axiomatically.

locale *life-table* =
fixes *l* :: *real* ⇒ *real* ($l′-- [101] 200)
assumes *l-0-pos*: *0* < *l 0*
and *l-neg-nil*: ⋀*x*. *x* ≤ *0* ⟹ *l x* = *l 0*
and *l-PInfty-0*: (*l* ⟶ *0*) *at-top*
and *l-antimono*: *antimono l*
and *l-right-continuous*: ⋀*x*. *continuous* (*at-right x*) *l*
begin

6.1 Basic Properties of Life Table

lemma *l-0-neq-0*[*simp*]: $*l-0* ≠ *0*
⟨*proof*⟩

lemma *l-nonneg*[*simp*]: $*l-x* ≥ *0* **for** *x*::*real*
⟨*proof*⟩

lemma *l-bounded*[*simp*]: $*l-x* ≤ $*l-0* **for** *x*::*real*
⟨*proof*⟩

lemma *l-measurable*[*measurable*, *simp*]: *l* ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *l-left-continuous-nonpos*: *continuous* (*at-left x*) *l* **if** *x* ≤ *0* **for** *x*::*real*
⟨*proof*⟩

lemma *l-integrable-Icc*: *set-integrable lborel* {*a*..*b*} *l* **for** *a b* :: *real*
⟨*proof*⟩

corollary *l-integrable-on-Icc*: *l integrable-on* {*a*..*b*} **for** *a b* :: *real*
⟨*proof*⟩

lemma *l-integrable-Icc-shift*: *set-integrable lborel* {*a*..*b*} (λ*t*. $*l*-(*x*+*t*)) **for** *a b x* :: *real*
⟨*proof*⟩

corollary *l-integrable-on-Icc-shift*: (λ*t*. $*l*-(*x*+*t*)) *integrable-on* {*a*..*b*} **for** *x a b* :: *real*
⟨*proof*⟩

lemma *l-normal-antimono*: *antimono* (λ*x*. $*l-x* / $*l-0*)
⟨*proof*⟩

lemma *compl-l-normal-right-continuous*: *continuous* (*at-right x*) ($\lambda x.$ *1* − \$*l-x* / \$*l-0*) **for** *x*::*real*
⟨*proof*⟩

lemma *compl-l-normal-NInfty-0*: (($\lambda x.$ *1* − \$*l-x* / \$*l-0*) ⟶ *0*) *at-bot*
⟨*proof*⟩

lemma *compl-l-normal-PInfty-1*: (($\lambda x.$ *1* − \$*l-x* / \$*l-0*) ⟶ *1*) *at-top*
⟨*proof*⟩

lemma *compl-l-real-distribution*: *real-distribution* (*interval-measure* ($\lambda x.$ *1* − \$*l-x* / \$*l-0*))
⟨*proof*⟩

definition *total* :: *real* ⇒ *real* (\$*T*′-- [*101*] *200*) **where** \$*T-x* ≡ *LBINT y*:{*x*..}. \$*l-y*
— the number of lives older than the ones aged *x*
— The parameter *x* must be nonnegative.

lemma *T-nonneg*[*simp*]: \$*T-x* ≥ *0* **for** *x*::*real*
⟨*proof*⟩

definition *total-finite* ≡ *set-integrable lborel* {*0*..} *l*

lemma *total-finite-iff-set-integrable-Ici*:
total-finite ⟷ *set-integrable lborel* {*x*..} *l* **for** *x*::*real*
⟨*proof*⟩

lemma *total-finite-iff-integrable-on-Ici*: *total-finite* ⟷ *l integrable-on* {*x*..} **for** *x*::*real*
⟨*proof*⟩

lemma *total-finite-iff-summable*: *total-finite* ⟷ *summable* ($\lambda k.$ \$*l*-(*x*+*k*)) **for** *x*::*real*
⟨*proof*⟩

lemma *T-tendsto-0*: (($\lambda x.$ \$*T-x*) ⟶ *0*) *at-top* **if** *total-finite*
⟨*proof*⟩

definition *lives* :: *real* ⇒ *real* ⇒ *real* (\$*L*′-{-&-} [*0*,*0*] *200*)
where \$*L*-{*n*&*x*} ≡ *LBINT y*:{*x*..*x*+*n*}. \$*l-y*
— the number of lives between ages *x* and *x*+*n*
— The parameter *x* must be nonnegative.
— The parameter *n* is usually nonnegative, but theoretically it can be negative.

abbreviation *lives-1* :: *real* ⇒ *real* (\$*L*′-- [*101*] *200*)
where \$*L-x* ≡ \$*L*-{*1*&*x*}

lemma *l-has-integral-L*: (*l has-integral* \$*L*-{*n*&*x*}) {*x*..*x*+*n*} **for** *x n* :: *real*

⟨*proof*⟩

lemma *L-neg-0*[*simp*]: $*L*-{*n*&*x*} = *0* **if** *n* < *0* **for** *x n* :: *real*
⟨*proof*⟩

lemma *L-nonneg*[*simp*]: $*L*-{*n*&*x*} ≥ *0* **for** *x n* :: *real*
⟨*proof*⟩

lemma *L-T*: $*L*-{*n*&*x*} = $*T*-*x* − $*T*-(*x*+*n*) **if** *total-finite n* ≥ *0* **for** *x n* :: *real*
⟨*proof*⟩

lemma *L-sums-T*: (λ*k*. $*L*-(*x*+*k*)) *sums* $*T*-*x* **if** *total-finite* **for** *x*::*real*
⟨*proof*⟩

definition *death* :: *real* ⇒ *real* ⇒ *real* ($*d*′-{-&-} [*0*,*0*] *200*)
where $*d*-{*t*&*x*} ≡ *max 0* ($*l*-*x* − $*l*-(*x*+*t*))
— the number of deaths between ages *x* and *x*+*t*
— The parameter *t* is usually nonnegative, but theoretically it can be negative.

abbreviation *death1* :: *real* ⇒ *real* ($*d*′-- [*101*] *200*)
where $*d*-*x* ≡ $*d*-{*1*&*x*}

lemma *death-def-nonneg*: $*d*-{*t*&*x*} = $*l*-*x* − $*l*-(*x*+*t*) **if** *t* ≥ *0* **for** *t x* :: *real*
⟨*proof*⟩

lemma *d-nonpos-0*: $*d*-{*t*&*x*} = *0* **if** *t* ≤ *0* **for** *t x* :: *real*
⟨*proof*⟩

corollary *d-0-0*: $*d*-{*0*&*x*} = *0* **for** *x*::*real*
⟨*proof*⟩

lemma *d-nonneg*[*simp*]: $*d*-{*t*&*x*} ≥ *0* **for** *t x* :: *real*
⟨*proof*⟩

lemma *dx-l*: $*d*-*x* = $*l*-*x* − $*l*-(*x*+*1*) **for** *x*::*real*
⟨*proof*⟩

lemma *sum-dx-l*: (∑ *k*<*n*. $*d*-(*x*+*k*)) = $*l*-*x* − $*l*-(*x*+*n*) **for** *x*::*real* **and** *n*::*nat*
⟨*proof*⟩

corollary *d-sums-l*: (λ*k*. $*d*-(*x*+*k*)) *sums* $*l*-*x* **for** *x*::*real*
⟨*proof*⟩

lemma *add-d*: $*d*-{*t*&*x*} + $*d*-{*t*′ & *x*+*t*} = $*d*-{*t*+*t*′ & *x*} **if** *t* ≥ *0 t*′ ≥ *0* **for** *t t*′ :: *real*
⟨*proof*⟩

definition *die-central* :: *real* ⇒ *real* ⇒ *real* ($*m*′-{-&-} [*0*,*0*] *200*)
where $*m*-{*n*&*x*} ≡ $*d*-{*n*&*x*} / $*L*-{*n*&*x*}

— central death rate

abbreviation *die-central-1* :: *real* ⇒ *real* ($*m*′-- [*101*] *200*)
where $*m-x* ≡ $*m-*{*1*&*x*}

6.2 Construction of Survival Model from Life Table

definition *life-table-measure* :: *real measure* (𝔐)
where 𝔐 ≡ *interval-measure* (λ*x*. *1* − $*l-x* / $*l-0*)

lemma *prob-space-actuary-MM*: *prob-space-actuary* 𝔐
⟨*proof*⟩

definition *survival-model-X* :: *real* ⇒ *real* (*X*) **where** *X* ≡ λ*x*. *x*

lemma *survival-model-MM-X*: *survival-model* 𝔐 *X*
⟨*proof*⟩

end

sublocale *life-table* ⊆ *survival-model* 𝔐 *X*
⟨*proof*⟩

context *life-table*
begin

interpretation *distrX-RD*: *real-distribution distr* 𝔐 *borel X*
⟨*proof*⟩

6.2.1 Relations between Life Table and Survival Function for *X*

lemma *ccdfX-l-normal*: *ccdf* (*distr* 𝔐 *borel X*) = (λ*x*. $*l-x* / $*l-0*)
⟨*proof*⟩

corollary *deriv-ccdfX-l*: *deriv* (*ccdf* (*distr* 𝔐 *borel X*)) *x* = *deriv l x* / $*l-0*
if *l differentiable at x* **for** *x*::*real*
⟨*proof*⟩

notation *death-pt* ($ψ)

lemma *l-0-equiv*: $*l-x* = *0* ⟷ *x* ≥ $ψ **for** *x*::*real*
⟨*proof*⟩

lemma *d-old-0*: $*d-*{*t*&*x*} = *0* **if** *x* ≥ $ψ *t* ≥ *0* **for** *x t* :: *real*
⟨*proof*⟩

lemma *d-l-equiv*: $*d-*{*t*&*x*} = $*l-x* ⟷ *x*+*t* ≥ $ψ **if** *t* ≥ *0* **for** *x t* :: *real*
⟨*proof*⟩

lemma *continuous-ccdfX-l*: *continuous F* (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) ⟷ *continuous F l*
for *F* :: *real filter*
⟨*proof*⟩

lemma *has-real-derivative-ccdfX-l*:
(*ccdf* (*distr* $\mathfrak{M}$ *borel X*) *has-real-derivative D*) (*at x*) ⟷
(*l has-real-derivative* $*l-0* ∗ *D*) (*at x*)
for *D x* :: *real*
⟨*proof*⟩

corollary *differentiable-ccdfX-l*:
ccdf (*distr* $\mathfrak{M}$ *borel X*) *differentiable* (*at x*) ⟷ *l differentiable* (*at x*)
for *D x* :: *real*
⟨*proof*⟩

lemma *PX-l-normal*: $\mathcal{P}$(*ξ in* $\mathfrak{M}$. *X ξ* > *x*) = $*l-x* / $*l-0* **for** *x*::*real*
⟨*proof*⟩

lemma *set-integrable-ccdfX-l*:
set-integrable lborel A (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) ⟷ *set-integrable lborel A l*
if *A* ∈ *sets lborel* **for** *A* :: *real set*
⟨*proof*⟩

corollary *integrable-ccdfX-l*: *integrable lborel* (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) ⟷ *integrable lborel l*
⟨*proof*⟩

lemma *integrable-on-ccdfX-l*:
ccdf (*distr* $\mathfrak{M}$ *borel X*) *integrable-on A* ⟷ *l integrable-on A* **for** *A* :: *real set*
⟨*proof*⟩

6.2.2 Relations between Life Table and Cumulative Distributive Function for *X*

lemma *cdfX-l-normal*: *cdf* (*distr* $\mathfrak{M}$ *borel X*) = (λ*x*. *1* − $*l-x* / $*l-0*) **for** *x*::*real*
⟨*proof*⟩

lemma *deriv-cdfX-l*: *deriv* (*cdf* (*distr* $\mathfrak{M}$ *borel X*)) *x* = − *deriv l x* / $*l-0*
if *l differentiable at x* **for** *x*::*real*
⟨*proof*⟩

lemma *continuous-cdfX-l*: *continuous F* (*cdf* (*distr* $\mathfrak{M}$ *borel X*)) ⟷ *continuous F l*
for *F* :: *real filter*
⟨*proof*⟩

lemma *has-real-derivative-cdfX-l*:
(*cdf* (*distr* $\mathfrak{M}$ *borel X*) *has-real-derivative D*) (*at x*) ⟷

(l *has-real-derivative* $-$ (*$l-0* $*$ D)) (*at* x)
for D x :: *real*
⟨*proof*⟩

lemma *differentiable-cdfX-l*:
cdf (*distr* $\mathfrak{M}$ *borel* X) *differentiable* (*at* x) $\longleftrightarrow$ l *differentiable* (*at* x) **for** D x :: *real*
⟨*proof*⟩

lemma *PX-compl-l-normal*: $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. X $\xi \leq x)$ = *1* $-$ *$l-x* / *$l-0* **for** *x::real*
⟨*proof*⟩

6.2.3 Relations between Life Table and Survival Function for $T(x)$

context
fixes *x::real*
assumes *x-lt-psi*[*simp*]: x < *$ψ*
begin

notation *futr-life* (T)

interpretation *alivex-PS*: *prob-space* $\mathfrak{M}$ ⇂ *alive* x
⟨*proof*⟩

interpretation *distrTx-RD*: *real-distribution distr* ($\mathfrak{M}$ ⇂ *alive* x) *borel* (T x) ⟨*proof*⟩

lemma *lx-neq-0*[*simp*]: *$l-x* $\neq$ *0*
⟨*proof*⟩

corollary *lx-pos*[*simp*]: *$l-x* > *0*
⟨*proof*⟩

lemma *ccdfTx-l-normal*: *ccdf* (*distr* ($\mathfrak{M}$ ⇂ *alive* x) *borel* (T x)) t = *$l-(x+t)* / *$l-x*
if $t \geq$ *0* **for** *t::real*
⟨*proof*⟩

lemma *deriv-ccdfTx-l*:
deriv (*ccdf* (*distr* ($\mathfrak{M}$ ⇂ *alive* x) *borel* (T x))) t = *deriv* (λt. *$l-(x+t)* / *$l-x*) t
if t > *0* l *differentiable at* (x+t) **for** *t::real*
⟨*proof*⟩

lemma *continuous-at-within-ccdfTx-l*:
continuous (*at* t *within* {*0..*}) (*ccdf* (*distr* ($\mathfrak{M}$ ⇂ *alive* x) *borel* (T x))) $\longleftrightarrow$
continuous (*at* (x+t) *within* {*x..*}) l
if $t \geq$ *0* **for** *t::real*
⟨*proof*⟩

lemma *isCont-ccdfTx-l*:

isCont (*ccdf* (*distr* ($\mathfrak{M}$ ↾ *alive x*) *borel* (*T x*))) *t* ⟷ *isCont l* (*x*+*t*) **if** *t* > *0* **for** *t*::*real*
⟨*proof*⟩

lemma *has-real-derivative-ccdfTx-l*:
(*ccdf* (*distr* ($\mathfrak{M}$ ↾ *alive x*) *borel* (*T x*)) *has-real-derivative D*) (*at t*) ⟷
(*l has-real-derivative* \$*l*-*x* ∗ *D*) (*at* (*x*+*t*))
if *t* > *0* **for** *t D* :: *real*
⟨*proof*⟩

lemma *differentiable-ccdfTx-l*:
ccdf (*distr* ($\mathfrak{M}$ ↾ *alive x*) *borel* (*T x*)) *differentiable at t* ⟷ *l differentiable* (*at* (*x*+*t*))
if *t* > *0* **for** *t*::*real*
⟨*proof*⟩

lemma *PTx-l-normal*: $\mathcal{P}$(*ξ in* $\mathfrak{M}$. *T x ξ* > *t* | *T x ξ* > *0*) = \$*l*-(*x*+*t*) / \$*l*-*x* **if** *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

6.2.4 Relations between Life Table and Cumulative Distributive Function for $T(x)$

lemma *cdfTx-compl-l-normal*: *cdf* (*distr* ($\mathfrak{M}$ ↾ *alive x*) *borel* (*T x*)) *t* = *1* − \$*l*-(*x*+*t*) / \$*l*-*x*
if *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *deriv-cdfTx-l*:
deriv (*cdf* (*distr* ($\mathfrak{M}$ ↾ *alive x*) *borel* (*T x*))) *t* = − *deriv* (*λt*. \$*l*-(*x*+*t*) / \$*l*-*x*) *t*
if *t* > *0 l differentiable at* (*x*+*t*) **for** *t*::*real*
⟨*proof*⟩

lemma *continuous-at-within-cdfTx-l*:
continuous (*at t within* {*0*..}) (*cdf* (*distr* ($\mathfrak{M}$ ↾ *alive x*) *borel* (*T x*))) ⟷
continuous (*at* (*x*+*t*) *within* {*x*..}) *l*
if *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *isCont-cdfTx-l*:
isCont (*cdf* (*distr* ($\mathfrak{M}$ ↾ *alive x*) *borel* (*T x*))) *t* ⟷ *isCont l* (*x*+*t*) **if** *t* > *0* **for** *t*::*real*
⟨*proof*⟩

lemma *has-real-derivative-cdfTx-l*:
(*cdf* (*distr* ($\mathfrak{M}$ ↾ *alive x*) *borel* (*T x*)) *has-real-derivative D*) (*at t*) ⟷
(*l has-real-derivative* − \$*l*-*x* ∗ *D*) (*at* (*x*+*t*))
if *t* > *0* **for** *t D* :: *real*
⟨*proof*⟩

lemma *differentiable-cdfTx-l*:
cdf (*distr* ($\mathfrak{M}$ $\downharpoonright$ *alive* x) *borel* (T x)) *differentiable at* $t \longleftrightarrow l$ *differentiable* (*at* ($x+t$))
if $t > 0$ **for** t::*real*
⟨*proof*⟩

lemma *PTx-compl-l-normal*: $\mathcal{P}(\xi$ *in* $\mathfrak{M}.$ $T\ x\ \xi \le t \mid T\ x\ \xi > 0) = 1-$ \$*l*-($x+t$) / \$*l*-x
if $t \ge 0$ **for** t::*real*
⟨*proof*⟩

6.2.5 Life Table and Actuarial Notations

notation *survive* (\$*p*′-{-&-} [*0*,*0*] *200*)
notation *survive-1* (\$*p*′-- [*101*] *200*)
notation *die* (\$*q*′-{-&-} [*0*,*0*] *200*)
notation *die-1* (\$*q*′-- [*101*] *200*)
notation *die-defer* (\$*q*′-{-|-&-} [*0*,*0*,*0*] *200*)
notation *die-defer-1* (\$*q*′-{-|&-} [*0*,*0*] *200*)
notation *life-expect* (\$*e*‘∘′-- [*101*] *200*)
notation *temp-life-expect* (\$*e*‘∘′-{-:-} [*0*,*0*] *200*)
notation *curt-life-expect* (\$*e*′-- [*101*] *200*)
notation *temp-curt-life-expect* (\$*e*′-{-:-} [*0*,*0*] *200*)

lemma *p-l*: \$*p*-{*t*&*x*} = \$*l*-($x+t$) / \$*l*-x **if** $t \ge 0$ **for** t::*real*
⟨*proof*⟩

corollary *p-1-l*: \$*p*-x = \$*l*-($x+1$) / \$*l*-x
⟨*proof*⟩

lemma *isCont-p-l*: *isCont* ($\lambda s.$ \$*p*-{*s*&*x*}) $t \longleftrightarrow$ *isCont* l ($x+t$) **if** $t > 0$ **for** t::*real*
⟨*proof*⟩

lemma *total-finite-iff-p-set-integrable-Ici*:
total-finite $\longleftrightarrow$ *set-integrable lborel* {*0*..} ($\lambda t.$ \$*p*-{*t*&*x*})
⟨*proof*⟩

lemma *p-PTx-ge-l-isCont*: \$*p*-{*t*&*x*} = $\mathcal{P}(\xi$ *in* $\mathfrak{M}.$ $T\ x\ \xi \ge t \mid T\ x\ \xi > 0)$
if *isCont* l ($x+t$) $t > 0$ **for** t::*real*
⟨*proof*⟩

lemma *q-defer-l*: \$*q*-{*f*|*t*&*x*} = (\$*l*-($x+f$) $-$ \$*l*-($x+f+t$)) / \$*l*-x **if** $f \ge 0$ $t \ge 0$ **for** f t :: *real*
⟨*proof*⟩

corollary *q-defer-d-l*: \$*q*-{*f*|*t*&*x*} = \$*d*-{*t* & *x*+*f*} / \$*l*-x **if** $f \ge 0$ $t \ge 0$ **for** f t :: *real*
⟨*proof*⟩

corollary *q-defer-1-d-l*: $\$q\text{-}\{f|\&x\} = \$d\text{-}(x+f)\ /\ \$l\text{-}x$ **if** $f \geq 0$ **for** f::*real*
⟨*proof*⟩

lemma *q-d-l*: $\$q\text{-}\{t\&x\} = \$d\text{-}\{t\&x\}\ /\ \$l\text{-}x$ **for** t::*real*
⟨*proof*⟩

corollary *q-1-d-l*: $\$q\text{-}x = \$d\text{-}x\ /\ \$l\text{-}x$
⟨*proof*⟩

lemma *LBINT-p-l*: $(LBINT\ t{:}A.\ \$p\text{-}\{t\&x\}) = (LBINT\ t{:}A.\ \$l\text{-}(x+t))\ /\ \$l\text{-}x$
if $A \subseteq \{0..\}$ $A \in$ *sets lborel* **for** A :: *real set*
— Note that $0 = 0$ holds when the integral diverges.
⟨*proof*⟩

corollary *e-LBINT-l*: $\$e`\circ\text{-}x = (LBINT\ t{:}\{0..\}.\ \$l\text{-}(x+t))\ /\ \$l\text{-}x$
— Note that $0 = 0$ holds when the integral diverges.
⟨*proof*⟩

corollary *e-LBINT-l-Icc*: $\$e`\circ\text{-}x = (LBINT\ t{:}\{0..n\}.\ \$l\text{-}(x+t))\ /\ \$l\text{-}x$ **if** $x+n \geq \$\psi$ **for** n::*real*
⟨*proof*⟩

lemma *temp-e-LBINT-l*: $\$e`\circ\text{-}\{x{:}n\} = (LBINT\ t{:}\{0..n\}.\ \$l\text{-}(x+t))\ /\ \$l\text{-}x$ **if** $n \geq 0$ **for** n::*real*
⟨*proof*⟩

lemma *integral-p-l*: *integral* A $(\lambda t.\ \$p\text{-}\{t\&x\}) = ($*integral* A $(\lambda t.\ \$l\text{-}(x+t)))\ /\ \$l\text{-}x$
if $A \subseteq \{0..\}$ $A \in$ *sets lborel* **for** A :: *real set*
— Note that $0 = 0$ holds when the integral diverges.
⟨*proof*⟩

corollary *e-integral-l*: $\$e`\circ\text{-}x =$ *integral* $\{0..\}$ $(\lambda t.\ \$l\text{-}(x+t))\ /\ \$l\text{-}x$
— Note that $0 = 0$ holds when the integral diverges.
⟨*proof*⟩

corollary *e-integral-l-Icc*:
$\$e`\circ\text{-}x =$ *integral* $\{0..n\}$ $(\lambda t.\ \$l\text{-}(x+t))\ /\ \$l\text{-}x$ **if** $x+n \geq \$\psi$ **for** n::*real*
⟨*proof*⟩

lemma *e-pos-total-finite*: $\$e`\circ\text{-}x > 0$ **if** *total-finite*
⟨*proof*⟩

lemma *temp-e-integral-l*:
$\$e`\circ\text{-}\{x{:}n\} =$ *integral* $\{0..n\}$ $(\lambda t.\ \$l\text{-}(x+t))\ /\ \$l\text{-}x$ **if** $n \geq 0$ **for** n::*real*
⟨*proof*⟩

lemma *curt-e-sum-l*: $\$e\text{-}x = (\sum k.\ \$l\text{-}(x+k+1))\ /\ \$l\text{-}x$ **if** *total-finite* $\bigwedge k{::}nat.$ *is-Cont* l $(x+k+1)$

⟨*proof*⟩

lemma *curt-e-sum-l-finite*: $\$e\text{-}x = (\sum k<n.\ \$l\text{-}(x+k+1))\ /\ \$l\text{-}x$
if $\bigwedge k::nat.\ k < n \Longrightarrow isCont\ l\ (x+k+1)\ x+n+1 > \$\psi$ **for** *n*::*nat*
⟨*proof*⟩

lemma *temp-curt-e-sum-p*: $\$e\text{-}\{x{:}n\} = (\sum k<n.\ \$l\text{-}(x+k+1))\ /\ \$l\text{-}x$
if $\bigwedge k::nat.\ k < n \Longrightarrow isCont\ l\ (x+k+1)$ **for** *n*::*nat*
⟨*proof*⟩

lemma *e-T-l*: $\$e`\circ\text{-}x = \$T\text{-}x\ /\ \$l\text{-}x$
⟨*proof*⟩

lemma *temp-e-L-l*: $\$e`\circ\text{-}\{x{:}n\} = \$L\text{-}\{n\&x\}\ /\ \$l\text{-}x$ **if** $n \geq 0$ **for** *n*::*real*
⟨*proof*⟩

lemma *m-q-e*: $\$m\text{-}\{n\&x\} = \$q\text{-}\{n\&x\}\ /\ \$e`\circ\text{-}\{x{:}n\}$ **if** $n \geq 0$ **for** *n*::*real*
⟨*proof*⟩

end

lemma *l-p*: $\$l\text{-}x\ /\ \$l\text{-}0 = \$p\text{-}\{x\&0\}$ **for** *x*::*real*
⟨*proof*⟩

lemma *e-p-e-total-finite*: $\$e`\circ\text{-}x = \$e`\circ\text{-}\{x{:}n\} + \$p\text{-}\{n\&x\} * \$e`\circ\text{-}(x+n)$
if *total-finite* $n \geq 0\ x+n < \$\psi$ **for** *x n* :: *real*
⟨*proof*⟩

proposition *x-ex-const-equiv-total-finite*: $x + \$e`\circ\text{-}x = y + \$e`\circ\text{-}y \longleftrightarrow \$q\text{-}\{y-x\&x\} = 0$
if *total-finite* $x \leq y\ y < \$\psi$ **for** *x y* :: *real*
⟨*proof*⟩

corollary *x-ex-const-iff-l-const*: $x + \$e`\circ\text{-}x = y + \$e`\circ\text{-}y \longleftrightarrow \$l\text{-}x = \$l\text{-}y$
if *total-finite* $x \leq y\ y < \$\psi$ **for** *x y* :: *real*
⟨*proof*⟩

end

6.3 Piecewise Differentiable Life Table

locale *smooth-life-table* = *life-table* +
assumes *l-piecewise-differentiable*[*simp*]: *l piecewise-differentiable-on UNIV*
begin

lemma *smooth-survival-function-MM-X*: *smooth-survival-function* $\mathfrak{M}$ *X*
⟨*proof*⟩

end

sublocale *smooth-life-table* $\subseteq$ *smooth-survival-function* $\mathfrak{M}$ *X*
⟨*proof*⟩

context *smooth-life-table*
begin

notation *force-mortal* (\$$\mu'$-- [*101*] *200*)

lemma *l-continuous*[*simp*]: *continuous-on UNIV l*
⟨*proof*⟩

lemma *l-nondifferentiable-finite-set*[*simp*]: *finite* {*x*. $\neg$ *l differentiable at x*}
⟨*proof*⟩

lemma *l-differentiable-borel-set*[*measurable*, *simp*]: {*x*. *l differentiable at x*} $\in$ *sets borel*
⟨*proof*⟩

lemma *l-differentiable-AE*: *AE x in lborel*. *l differentiable at x*
⟨*proof*⟩

lemma *deriv-l-measurable*[*measurable*]: *deriv l* $\in$ *borel-measurable borel*
⟨*proof*⟩

lemma *pdfX-l-normal*:
pdfX x = (*if l differentiable at x then* $-$ *deriv l x* / \$*l-0 else 0*) **for** *x*::*real*
⟨*proof*⟩

lemma *mu-deriv-l*: \$$\mu$-*x* = $-$ *deriv l x* / \$*l-x* **if** *l differentiable at x* **for** *x*::*real*
⟨*proof*⟩

lemma *mu-nonneg-differentiable-l*: \$$\mu$-*x* $\geq$ *0* **if** *l differentiable at x* **for** *x*::*real*
⟨*proof*⟩

lemma *mu-deriv-ln-l*:
\$$\mu$-*x* = $-$ *deriv* (λ*x*. *ln* (\$*l-x*)) *x* **if** *l differentiable at x x* < \$$\psi$ **for** *x*::*real*
⟨*proof*⟩

lemma *deriv-l-shift*: *deriv l* (*x*+*t*) = *deriv* (λ*t*. \$*l*-(*x*+*t*)) *t*
if *l differentiable at* (*x*+*t*) **for** *x t* :: *real*
⟨*proof*⟩

context
fixes *x*::*real*
assumes *x-lt-psi*[*simp*]: *x* < \$$\psi$
begin

lemma *p-mu-l*: \$*p*-{*t*&*x*} $*$ \$$\mu$-(*x*+*t*) = $-$ *deriv l* (*x*+*t*) / \$*l-x*

if *l differentiable at* (*x*+*t*) *t* > *0* *x*+*t* < $ψ **for** *t*::*real*
⟨*proof*⟩

lemma *p-mu-l-AE*: *AE s in lborel*. *0* < *s* ∧ *x*+*s* < $ψ ⟶ $*p*-{*s*&*x*} ∗ $μ-(*x*+*s*) = − *deriv l* (*x*+*s*) / $*l*-*x*
⟨*proof*⟩

lemma *LBINT-l-mu-q*: (*LBINT s*:{*f*<..*f*+*t*}. $*l*-(*x*+*s*) ∗ $μ-(*x*+*s*)) / $*l*-*x* = $*q*-{*f*|*t*&*x*}
if *t* ≥ *0* *f* ≥ *0* **for** *t f* :: *real*
⟨*proof*⟩

lemma *set-integrable-l-mu*: *set-integrable lborel* {*f*<..*f*+*t*} (λ*s*. $*l*-(*x*+*s*) ∗ $μ-(*x*+*s*))
if *t* ≥ *0* *f* ≥ *0* **for** *t f* :: *real*
⟨*proof*⟩

lemma *l-mu-has-integral-q-defer*:
((λ*s*. $*l*-(*x*+*s*) ∗ $μ-(*x*+*s*) / $*l*-*x*) *has-integral* $*q*-{*f*|*t*&*x*}) {*f*..*f*+*t*}
if *t* ≥ *0* *f* ≥ *0* **for** *t f* :: *real*
⟨*proof*⟩

corollary *l-mu-has-integral-q*:
((λ*s*. $*l*-(*x*+*s*) ∗ $μ-(*x*+*s*) / $*l*-*x*) *has-integral* $*q*-{*t*&*x*}) {*0*..*t*} **if** *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *l-mu-has-integral-d*:
((λ*s*. $*l*-(*x*+*s*) ∗ $μ-(*x*+*s*)) *has-integral* $*d*-{*t* & *x*+*f*}) {*f*..*f*+*t*}
if *t* ≥ *0* *f* ≥ *0* **for** *t f* :: *real*
⟨*proof*⟩

corollary *l-mu-has-integral-d-1*:
((λ*s*. $*l*-(*x*+*s*) ∗ $μ-(*x*+*s*)) *has-integral* $*d*-(*x*+*f*)) {*f*..*f*+*1*} **if** *t* ≥ *0* *f* ≥ *0* **for** *t f* :: *real*
⟨*proof*⟩

lemma *e-LBINT-l*: $*e*'○-*x* = (*LBINT s*:{*0*..}. $*l*-(*x*+*s*) ∗ $μ-(*x*+*s*) ∗ *s*) / $*l*-*x*
— Note that *0* = *0* holds when the life expectation diverges.
⟨*proof*⟩

lemma *e-integral-l*: $*e*'○-*x* = *integral* {*0*..} (λ*s*. $*l*-(*x*+*s*) ∗ $μ-(*x*+*s*) ∗ *s*) / $*l*-*x*
— Note that *0* = *0* holds when the life expectation diverges.
⟨*proof*⟩

lemma *m-LBINT-p-mu*: $*m*-{*n*&*x*} = (*LBINT t*:{*0*<..*n*}. $*p*-{*t*&*x*} ∗ $μ-(*x*+*t*)) / (*LBINT t*:{*0*..*n*}. $*p*-{*t*&*x*})
if *n* ≥ *0* **for** *n*::*real*
⟨*proof*⟩

lemma *m-integral-p-mu*:
$*m*-{*n*&*x*} = *integral* {*0*..*n*} (λ*t*. $*p*-{*t*&*x*} ∗ $μ-(*x*+*t*)) / *integral* {*0*..*n*} (λ*t*.

$\$p\text{-}\{t\&x\})$
if $n \geq 0$ **for** $n::real$
⟨*proof*⟩

end

lemma *deriv-x-p-mu-l*: $deriv\ (\lambda y.\ \$p\text{-}\{t\&y\})\ x = \$p\text{-}\{t\&x\} * (\$\mu\text{-}x - \$\mu\text{-}(x+t))$
if *l differentiable at x l differentiable at* $(x+t)$ $t \geq 0$ $x < \$\psi$ **for** $x\ t :: real$
⟨*proof*⟩

lemma *e-has-derivative-mu-e-l*: $((\lambda x.\ \$e`\circ\text{-}x)$ *has-real-derivative* $(\$\mu\text{-}x * \$e`\circ\text{-}x - 1))$ $(at\ x)$
if *total-finite l differentiable at x* $x \in \{a<..<b\}$ $b \leq \$\psi$ **for** $a\ b\ x :: real$
⟨*proof*⟩

corollary *e-has-derivative-mu-e-l'*: $((\lambda x.\ \$e`\circ\text{-}x)$ *has-real-derivative* $(\$\mu\text{-}x * \$e`\circ\text{-}x - 1))$ $(at\ x)$
if *total-finite l differentiable at x* $x \in \{a<..<b\}$ $b \leq \$\psi$ **for** $a\ b\ x :: real$
⟨*proof*⟩

context
fixes $x::real$
assumes *x-lt-psi*[*simp*]: $x < \$\psi$
begin

lemma *curt-e-sum-l-smooth*: $\$e\text{-}x = (\sum k.\ \$l\text{-}(x+k+1))\ /\ \$l\text{-}x$ **if** *total-finite*
⟨*proof*⟩

lemma *curt-e-sum-l-finite-smooth*: $\$e\text{-}x = (\sum k<n.\ \$l\text{-}(x+k+1))\ /\ \$l\text{-}x$ **if** $x+n+1 > \$\psi$ **for** $n::nat$
⟨*proof*⟩

lemma *temp-curt-e-sum-l-smooth*: $\$e\text{-}\{x{:}n\} = (\sum k<n.\ \$l\text{-}(x+k+1))\ /\ \$l\text{-}x$ **for** $n::nat$
⟨*proof*⟩

end

end

6.4 Interpolations

context *life-table*
begin

definition *linear-interpolation* ≡
$\forall (x::nat)(t::real).\ 0 \leq t \wedge t \leq 1 \longrightarrow \$l\text{-}(x+t) = (1-t)*\$l\text{-}x + t*\$l\text{-}(x+1)$

lemma *linear-l*: $\$l\text{-}(x+t) = (1-t)*\$l\text{-}x + t*\$l\text{-}(x+1)$

if *linear-interpolation* $0 \leq t$ $t \leq 1$ **for** *x::nat* **and** *t::real*
⟨*proof*⟩

lemma *linear-l-d*: $\$l\text{-}(x+t) = \$l\text{-}x - t*\$d\text{-}x$
if *linear-interpolation* $0 \leq t$ $t \leq 1$ **for** *x::nat* **and** *t::real*
⟨*proof*⟩

lemma *linear-p-q*: $\$p\text{-}\{t\&x\} = 1 - t*\$q\text{-}x$
if *linear-interpolation* $0 \leq t$ $t \leq 1$ $x < \$\psi$ **for** *x::nat* **and** *t::real*
⟨*proof*⟩

lemma *linear-q*: $\$q\text{-}\{t\&x\} = t*\$q\text{-}x$
if *linear-interpolation* $0 \leq t$ $t \leq 1$ $x < \$\psi$ **for** *x::nat* **and** *t::real*
⟨*proof*⟩

lemma *linear-L-l-d*: $\$L\text{-}x = \$l\text{-}x - \$d\text{-}x\ /\ 2$ **if** *linear-interpolation* **for** *x::nat*
⟨*proof*⟩

lemma *linear-L-l-d'*: $\$L\text{-}x = \$l\text{-}(x+1) + \$d\text{-}x\ /\ 2$ **if** *linear-interpolation* **for** *x::nat*
⟨*proof*⟩

lemma *linear-l-continuous*: *continuous-on UNIV l* **if** *linear-interpolation*
⟨*proof*⟩

lemma *linear-l-sums-T-l*: $(\lambda k.\ \$l\text{-}(x + Suc\ k))$ *sums* $(\$T\text{-}x - \$l\text{-}x\ /\ 2)$
if *linear-interpolation total-finite* **for** *x::nat*
⟨*proof*⟩

corollary *linear-T-suminf-l*: $\$T\text{-}x = (\sum k.\ \$l\text{-}(x+k+1)) + \$l\text{-}x\ /\ 2$
if *linear-interpolation total-finite* **for** *x::nat*
⟨*proof*⟩

lemma *linear-mx-q*: $\$m\text{-}x = \$q\text{-}x\ /\ (1 - \$q\text{-}x\ /\ 2)$ **if** *linear-interpolation* $x < \$\psi$ **for** *x::nat*
⟨*proof*⟩

lemma *linear-e-curt-e*: $\$e\text{`}\circ\text{-}x = \$e\text{-}x + 1/2$
if *linear-interpolation total-finite* $x < \$\psi$ **for** *x::nat*
⟨*proof*⟩

end

context *smooth-life-table*
begin

lemma *linear-l-has-derivative-at-frac*:
$((\lambda s.\ \$l\text{-}(x+s))$ *has-real-derivative* $- \$d\text{-}x)$ *(at t)*
if *linear-interpolation* $0 < t$ $t < 1$ **for** *x::nat* **and** *t::real*
⟨*proof*⟩

lemma *linear-l-has-derivative-at-frac′*:
(*l has-real-derivative* − $*d-x*) (*at y*)
if *linear-interpolation* $x < y$ $y < x+1$ **for** *x*::*nat* **and** *y*::*real*
⟨*proof*⟩

lemma *linear-l-differentiable-on-frac*:
l differentiable-on $\{x<..<x+1\}$ **if** *linear-interpolation* **for** *x*::*nat*
⟨*proof*⟩

lemma *linear-l-has-right-derivative-at-nat*:
(*l has-real-derivative* − $*d-x*) (*at-right x*) **if** *linear-interpolation* **for** *x*::*nat*
⟨*proof*⟩

lemma *linear-l-has-left-derivative-at-nat*:
(*l has-real-derivative* − $*d*-(*real x* − *1*)) (*at-left x*) **if** *linear-interpolation* **for** *x*::*nat*
⟨*proof*⟩

lemma *linear-l-has-derivative-at-nat-iff-d*:
(*l has-real-derivative* − $*d-x*) (*at x*) $\longleftrightarrow$ $*d-x* = $*d*-(*real x* − *1*)
if *linear-interpolation* **for** *x*::*nat*
⟨*proof*⟩

lemma *linear-l-differentiable-at-nat-iff-d*:
l differentiable at x $\longleftrightarrow$ $*d-x* = $*d*-(*real x* − *1*)
if *linear-interpolation* **for** *x*::*nat*
⟨*proof*⟩

lemma *linear-l-limited*: $$\psi$ $< \infty$ **if** *linear-interpolation*
⟨*proof*⟩

lemma *linear-mu-q*: $$\mu$-(*x*+*t*) = $*q-x* / (*1* − *t*∗$*q-x*)
if *linear-interpolation l differentiable at* (*x*+*t*) $0 < t$ $t < 1$ *x*+*t* < $$\psi$
for *x*::*nat* **and** *t*::*real*
⟨*proof*⟩

definition *exponential-interpolation* ≡
∀ (*x*::*nat*)(*t*::*real*). *x*+*1* < $$\psi$ ⟶ $0 \le t \wedge t < 1$ ⟶ $$\mu$-(*x*+*t*) = $$\mu$-*x*
— Without *x*+*1* < $$\psi$, the smooth life table could not be limited.

lemma *exponential-mu*: $$\mu$-(*x*+*t*) = $$\mu$-*x*
if *exponential-interpolation* *x*+*1* < $$\psi$ $0 \le t$ $t < 1$ **for** *x*::*nat* **and** *t*::*real*
⟨*proof*⟩

corollary *exponential-mu′*: $$\mu$-*y* = $$\mu$-*x*
if *exponential-interpolation* $x \le y$ $y < x+1$ *x*+*1* < $$\psi$ **for** *x*::*nat* **and** *y*::*real*
⟨*proof*⟩

lemma *exponential-integral-mu*: $integral\ \{x..<x+t\}\ (\lambda y.\ \$\mu\text{-}y) = \$\mu\text{-}x * t$
if $exponential\text{-}interpolation\ x+1 < \$\psi\ \ 0 \le t\ \ t \le 1$ **for** $x::nat$ **and** $t::real$
⟨*proof*⟩

lemma *exponential-p-mu*: $\$p\text{-}x = exp\ (-\$\mu\text{-}x)$ **if** $exponential\text{-}interpolation\ x+1 < \$\psi$ **for** $x::nat$
⟨*proof*⟩

corollary *exponential-mu-p*: $\$\mu\text{-}x = -\ ln\ (\$p\text{-}x)$ **if** $exponential\text{-}interpolation\ x+1 < \$\psi$ **for** $x::nat$
⟨*proof*⟩

corollary *exponential-mu-xt-p*: $\$\mu\text{-}(x+t) = -\ ln\ (\$p\text{-}x)$
if $exponential\text{-}interpolation\ x+1 < \$\psi\ \ 0 \le t\ \ t < 1$ **for** $x::nat$ **and** $t::real$
⟨*proof*⟩

corollary *exponential-q-mu*: $\$q\text{-}x = 1 - exp\ (-\$\mu\text{-}x)$
if $exponential\text{-}interpolation\ x+1 < \$\psi$ **for** $x::nat$
⟨*proof*⟩

lemma *exponential-p*: $\$p\text{-}\{t\&x\} = (\$p\text{-}x).\hat{}\,t$
if $exponential\text{-}interpolation\ x+1 < \$\psi\ \ 0 \le t\ \ t \le 1$ **for** $x::nat$ **and** $t::real$
⟨*proof*⟩

lemma *exponential-q*: $\$q\text{-}\{t\&x\} = 1 - (1 - \$q\text{-}x).\hat{}\,t$
if $exponential\text{-}interpolation\ x+1 < \$\psi\ \ 0 \le t\ \ t \le 1$ **for** $x::nat$ **and** $t::real$
⟨*proof*⟩

lemma *exponential-l-p*: $\$l\text{-}(x+t) = \$l\text{-}x * (\$p\text{-}x).\hat{}\,t$
if $exponential\text{-}interpolation\ x+1 < \$\psi\ \ 0 \le t\ \ t \le 1$ **for** $x::nat$ **and** $t::real$
⟨*proof*⟩

lemma *exponential-l-has-derivative-at-frac*:
$((\lambda s.\ \$l\text{-}(x+s))\ has\text{-}real\text{-}derivative\ (-\ \$l\text{-}x * \$\mu\text{-}x * (\$p\text{-}x).\hat{}\,t))\ (at\ t)$
if $exponential\text{-}interpolation\ x+1 < \$\psi\ \ 0 < t\ \ t < 1$ **for** $x::nat$ **and** $t::real$
⟨*proof*⟩

lemma *exponential-l-has-derivative-at-frac'*:
$(l\ has\text{-}real\text{-}derivative\ (-\ \$l\text{-}x * \$\mu\text{-}x * (\$p\text{-}x).\hat{}\,(y-x)))\ (at\ y)$
if $exponential\text{-}interpolation\ x+1 < \$\psi\ \ x < y\ \ y < x+1$ **for** $x::nat$ **and** $y::real$
⟨*proof*⟩

lemma *exponential-l-differentiable-on-frac*:
$l\ differentiable\text{-}on\ \{x<..<x+1\}$ **if** $exponential\text{-}interpolation\ x+1 < \$\psi$ **for** $x::nat$
⟨*proof*⟩

lemma *exponential-l-has-right-derivative-at-nat*:
$(l\ has\text{-}real\text{-}derivative\ (-\ \$l\text{-}x * \$\mu\text{-}x))\ (at\text{-}right\ x)$
if $exponential\text{-}interpolation\ x+1 < \$\psi$ **for** $x::nat$

⟨*proof*⟩

lemma *exponential-l-has-left-derivative-at-nat*:
  (*l has-real-derivative* (− $\$l$-*x* ∗ $\$\mu$-(*real x* − *1*))) (*at-left x*)
  if *exponential-interpolation* *x* < $\$\psi$ **for** *x*::*nat*
⟨*proof*⟩

lemma *exponential-l-has-derivative-at-nat-iff-mu*:
  (*l has-real-derivative* (− $\$l$-*x* ∗ $\$\mu$-*x*)) (*at x*) ⟷ $\$\mu$-*x* = $\$\mu$-(*real x* − *1*)
  if *exponential-interpolation* *x*+*1* < $\$\psi$ **for** *x*::*nat*
⟨*proof*⟩

lemma *exponential-l-differentiable-at-nat-iff-mu*:
  l differentiable at x ⟷ $\$\mu$-*x* = $\$\mu$-(*real x* − *1*)
  if *exponential-interpolation* *x*+*1* < $\$\psi$ **for** *x*::*nat*
⟨*proof*⟩

lemma *exponential-L-d-mu*: $\$L$-*x* = $\$d$-*x* / $\$\mu$-*x*
  if *exponential-interpolation* $\$\mu$-*x* ≠ *0* *x*+*1* < $\$\psi$ **for** *x*::*nat*
⟨*proof*⟩

lemma *exponential-mx-mu*: $\$m$-*x* = $\$\mu$-*x* **if** *exponential-interpolation* *x*+*1* < $\$\psi$
for *x*::*nat*
⟨*proof*⟩

lemma *exponential-d-mu-sums-T*: (λ*k*. $\$d$-(*x*+*k*) / $\$\mu$-(*x*+*k*)) *sums* $\$T$-*x*
  if *exponential-interpolation total-finite* ⋀*k*::*nat*. $\$\mu$-(*x*+*k*) ≠ *0* **for** *x*::*nat*
⟨*proof*⟩

lemma *exponential-e-d-l-mu*: (λ*k*. $\$d$-(*x*+*k*) / ($\$l$-*x* ∗ $\$\mu$-(*x*+*k*))) *sums* $\$e$'∘-*x*
  if *exponential-interpolation total-finite* ⋀*k*::*nat*. $\$\mu$-(*x*+*k*) ≠ *0* **for** *x*::*nat*
⟨*proof*⟩

end

6.5 Limited Life Table

locale *limited-life-table* = *life-table* +
  assumes *l-limited*: ∃ *x*::*real*. $\$l$-*x* = *0*
begin

lemma *limited-survival-function-MM-X*: *limited-survival-function* 𝔐 *X*
⟨*proof*⟩

end

sublocale *limited-life-table* ⊆ *limited-survival-function* 𝔐 *X*
  ⟨*proof*⟩

context *limited-life-table*
begin

notation *ult-age* ($\$\omega$)

lemma *l-omega-0*: $\$l\text{-}\$\omega = 0$
⟨*proof*⟩

lemma *l-0-equiv-nat*: $\$l\text{-}x = 0 \longleftrightarrow x \geq \$\omega$ **for** *x::nat*
⟨*proof*⟩

lemma *d-l-equiv-nat*: $\$d\text{-}\{t\&x\} = \$l\text{-}x \longleftrightarrow x+t \geq \$\omega$ **if** $t \geq 0$ **for** *x t :: nat*
⟨*proof*⟩

corollary *d-1-omega-l*: $\$d\text{-}(\$\omega-1) = \$l\text{-}(\$\omega-1)$
⟨*proof*⟩

lemma *limited-life-table-imp-total-finite*: *total-finite*
⟨*proof*⟩

context
fixes *x::nat*
assumes *x-lt-omega*[*simp*]: $x < \$\omega$
begin

lemma *curt-e-sum-l-finite-nat*: $\$e\text{-}x = (\sum k{<}n.\ \$l\text{-}(x+k+1))\ /\ \$l\text{-}x$
if $\bigwedge k{::}nat.\ k < n \Longrightarrow isCont\ l\ (x+k+1)$ $x+n \geq \$\omega$ **for** *n::nat*
⟨*proof*⟩

end

end

end
theory *Examples*
imports *Life-Table*
begin

7 Examples

The following lemma is a verification of the solution to the multiple choice question No. 3 of Exam LTAM Spring 2022 by Society of Actuaries.

context *smooth-survival-function*
begin

lemma *SoA-LTAM-2022-Spring-MCQ-No3*:
assumes $\bigwedge x{::}real.\ 0 \leq x \Longrightarrow x \leq 100 \Longrightarrow ccdf\ (distr\ \mathfrak{M}\ borel\ X)\ x = (1 - 0.01{*}x).\hat{}\,0.5$

shows $|1000*\$\mu\text{-}25 - 6.7| < 0.05$
⟨*proof*⟩

end

The following lemma is a verification of the solution to the problem No. 2. (1)-1 of Life Insurance Mathematics 2016 by the Institute of Actuaries of Japan, slightly modified; see the remark below.

context *smooth-life-table*
begin

lemma *IAJ-Life-Insurance-Math-2016-2-1-1*:
fixes *a b* :: *real*
assumes $-1 < a$ $a < 0$ $0 < b$ $-b/a \leq \$\psi$ **and**
total-finite **and**
$\bigwedge x.\ 0 < x \Longrightarrow x < -b/a \Longrightarrow l$ *differentiable at x* **and**
$\bigwedge x.\ 0 \leq x \Longrightarrow x < -b/a \Longrightarrow \$e`\circ\text{-}x = a*x + b$
shows $\bigwedge x.\ 0 \leq x \Longrightarrow x < -b/a \Longrightarrow \$l\text{-}x = \$l\text{-}0 * (b\ /\ (a*x + b)).\widehat{}((a+1)/a)$
⟨*proof*⟩

REMARK. The original problem lacks the following hypotheses: (i) $0 < b$, (ii) $-b/a \leq \$\psi$, (iii) $\forall x.\ 0 < x < -b/a \longrightarrow l$ *differentiable at x*, (iv) $\forall x.\ 0 \leq x < -b/a \longrightarrow l$ *integrable-on* $\{x..\}$. Moreover, the hypothesis $\forall x.\ 0 \leq x < -b/a$ is originally $\forall x.\ 0 \leq x \leq -b/a$.

end

end