FRACTIONAL POISSON PROCESSES

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Abstract. In this review paper we present some fractional versions of the Poisson process. The one-dimensional distributions $p_k^\nu(t) = \Pr\{N^\nu(t) = k\}$, $k \geq 0$, are given and the corresponding governing fractional equations introduced and discussed. The last section of this note is devoted to the fractional birth process in the linear and non-linear cases.

1 Introduction The Poisson processes (homogeneous, non-homogeneous, multidimensional) play a basic role in many fields of applied probability. Therefore its possible extensions are of interest and recently some fractional versions of the Poisson process appeared in the literature.

The fractional Poisson process is useful for modelling flows of data displaying long memory (for example in network traffic, see Uchaikin et al. [14]). In some problems the intertime between successive events has non-exponential, heavy-tailed structure as in the fractional Poisson processes.

The fractional pure birth process is suitable to describe rapidly expanding epidemies (see for example Cahoy and Polito [6]).

2 Time-fractional Poisson process A fractional Poisson process of parameter $\nu \in (0,1)$ regarded as a renewal process with intertime between successive events T_k^ν, $k \geq 0$, possessing distribution

$$\Pr\{T_k^\nu \geq t\} = E_{\nu,1}(-\lambda t^\nu), \qquad t > 0, \tag{2.1}$$

and density

$$\Pr\{T_k^\nu \in dt\} = \nu\lambda t^{\nu-1} E_{\nu,\nu}(-\lambda t^\nu)\, dt, \tag{2.2}$$

has been introduced in Mainardi et al. [9] and also in Beghin and Orsingher [2]. The parameter λ appearing in (2.1) and (2.2) is a positive real number. In (2.1) and (2.2) appears the two-parameter Mittag-Leffler function defined as

$$E_{\nu,\mu}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\nu k + \mu)}, \qquad x \in \mathbb{R}, \nu > 0, \mu > 0. \tag{2.3}$$

On the basis of the renewal structure of the above defined model, it has been shown that the associated counting process $N^\nu(t)$, $t > 0$, possesses distribution

$$p_k(t) = \Pr\{N^\nu(t) = k\} = \frac{(\lambda t^\nu)^k}{k!} \sum_{r=0}^{\infty} \frac{(r+k)!}{r!} \frac{(-\lambda t^\nu)^r}{\Gamma(\nu(k+r)+1)}, \qquad k \geq 0. \tag{2.4}$$

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The distribution (2.4) has also been obtained by considering the fractional equation

$$\frac{d^\nu}{dt^\nu} p_k(t) = -\lambda p_k(t) + \lambda p_{k-1}(t), \qquad k \geq 0, \tag{2.5}$$

subject to the initial condition

$$p_k(0) = \begin{cases} 0, & k \geq 1, \\ 1, & k = 0, \end{cases} \tag{2.6}$$

for the state probabilities p_k with $p_{-1}(t) = 0$. The fractional derivative appearing in (2.5) is the Dzerbayshan-Caputo derivative (as considered in Beghin and Orsingher [2])

$$\frac{d^\nu}{dt^\nu} u(t) = \begin{cases} \frac{1}{\Gamma(m-\nu)} \int_0^t \frac{1}{(t-s)^{1+\nu-m}} \frac{d^m}{ds^m} u(s)\, ds, & m-1 < \nu < m, \\ \frac{d^m}{dt^m} u(t), & \nu = m, \end{cases} \tag{2.7}$$

while in Laskin [7] it has the form of the classical Riemann-Liouville derivative.

The relationship between the classical homogeneous Poisson process $N(t)$, $t > 0$, and the time-fractional Poisson process $N^\nu(t)$, $t > 0$, is represented by the subordination

$$N^\nu(t) \overset{law}{=} N\left(T_{2\nu}(t)\right), \qquad t > 0. \tag{2.8}$$

Formula (2.8) shows that the time-fractional Poisson process can be regarded as a homogeneous Poisson process at a random time $T_{2\nu}(t)$, $t > 0$, whose distribution

$$\Pr\left\{T_{2\nu}(t) \in ds\right\} = u(s,t)\, ds \tag{2.9}$$

is obtained by folding the fundamental solution to the fractional diffusion equation

$$\frac{\partial^{2\nu}}{\partial t^{2\nu}} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t), \qquad u(x,0) = \delta(x) \tag{2.10}$$

$$\text{with } u_t(x,0) = 0 \text{ for } \nu \in \left(\frac{1}{2}, 1\right).$$

In view of this we can rewrite the distribution of (2.4) as

$$\Pr\left\{N^\nu(t) = k\right\} = \int_0^\infty e^{-\lambda y} \frac{(\lambda y)^k}{k!} \frac{1}{t^\nu} W_{-\nu,1-\nu}\left(-y t^{-\nu}\right) dy \tag{2.11}$$

where

$$W_{\alpha,\beta}(x) = \sum_{k=0}^\infty \frac{x^k}{k!\Gamma\left(\alpha k + \beta\right)}, \qquad x \in \mathbb{R}, \alpha > -1, \beta > 0, \tag{2.12}$$

is the Wright function.

3 Space-fractional Poisson process A space-fractional Poisson process $\mathcal{N}^\alpha(t)$, $t > 0$, of parameter $\alpha \in (0,1]$, has been introduced recently in Orsingher and Polito [12] by considering a fractionalized version of the equation governing the state probabilities

$$p_k^\alpha(t) = \Pr\left\{\mathcal{N}^\alpha(t) = k\right\}, \qquad k \geq 0, \tag{3.1}$$

that is

$$\frac{d}{dt}p_k^{\alpha}(t) = -\lambda^{\alpha}(1-B)^{\alpha}p_k^{\alpha}(t), \qquad k \geq 0. \tag{3.2}$$

The fractional difference operator

$$\Delta^{\alpha} = (1-B)^{\alpha} \tag{3.3}$$

involves the shift operator B whose main properties are

$$\begin{cases} Bp_k^{\alpha}(t) = p_{k-1}^{\alpha}(t) \\ B^r p_k^{\alpha}(t) = B^{r-1}\left(Bp_k^{\alpha}(t)\right) = B^{r-1}p_{k-1}^{\alpha}(t), \end{cases} \tag{3.4}$$

for $k \geq 1$, $r \in \mathbb{N}$ and $Bp_0^{\alpha}(t) = 0$. The equation (3.2) can be written as

$$\frac{d}{dt}p_k^{\alpha}(t) = -\lambda^{\alpha}\sum_{r=0}^{k}\frac{\Gamma(\alpha+1)}{r!\,\Gamma(\alpha+1-r)}(-1)^r p_{k-r}^{\alpha}(t), \tag{3.5}$$

and shows the dependency of $p_k^{\alpha}(t)$ from all $p_{k-r}^{\alpha}(t)$, $0 \leq r \leq k$, $t > 0$. By resorting to the probability generating function

$$G_{\alpha}(u,t) = \sum_{k=0}^{\infty}u^k p_k^{\alpha}(t), \qquad |u| < 1, t > 0, \tag{3.6}$$

we are able to obtain the equation

$$\frac{\partial}{\partial t}G_{\alpha}(u,t) = -\lambda^{\alpha}(1-u)^{\alpha}G_{\alpha}(u,t) \tag{3.7}$$

subject to the initial condition

$$G_{\alpha}(u,0) = 1. \tag{3.8}$$

From (3.7) it is easy to show that

$$G_{\alpha}(u,t) = e^{-\lambda^{\alpha}t(1-u)^{\alpha}} \tag{3.9}$$

and thus

$$\Pr\{\mathcal{N}^{\alpha}(t) = k\} = \frac{(-1)^k}{k!}\sum_{r=0}^{\infty}\frac{(-\lambda^{\alpha}t)^r}{r!}\frac{\Gamma(\alpha r+1)}{\Gamma(\alpha r+1-k)}, \qquad k \geq 0, t > 0. \tag{3.10}$$

The exponential form (3.9) of the probability generating function of the space fractional Poisson process suggests that it mantains the independence of increments of the homogeneous Poisson process. The probability generating function of $\mathcal{N}^{\alpha}(t)$, $t > 0$, can be represented as

$$G_{\alpha}(u,t) = e^{-\lambda^{\alpha}(1-u)^{\alpha}t} = \Pr\left\{\min_{1\leq k\leq N(t)} X_k^{\frac{1}{\alpha}} \geq 1-u\right\}, \qquad 0 < u < 1, \tag{3.11}$$

where X_k, $k \geq 0$, are independent r.v.'s uniform on $[0,1]$ and $N(t)$, $t > 0$, is the homogeneous Poisson process of rate λ^{α}. The space fractionality is therefore introduced by the exponent $\frac{1}{\alpha}$ figuring in $X_k^{\frac{1}{\alpha}}$.

For $\alpha = 1$ we recover the classical Poisson process. The waiting time of the k-th event

$$\tau_k^\alpha = \inf\{t : \mathcal{N}^\alpha(t) = k\} \tag{3.12}$$

has distribution

$$\begin{aligned}\Pr\{\tau_k^\alpha < t\} &= \Pr\{\mathcal{N}^\alpha(t) \geq k\} \\ &= \sum_{m=k}^{\infty} \frac{(-1)^m}{m!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha t)^r}{r!} \frac{\Gamma(\alpha r + 1)}{\Gamma(\alpha r + 1 - m)}, \qquad t > 0.\end{aligned} \tag{3.13}$$

For $\alpha = 1$ we can extract from (3.13) the Erlang distribution

$$\Pr\left\{T_k^1 < t\right\} = e^{-\lambda t} \sum_{m=k}^{\infty} \frac{(\lambda t)^m}{m!}, \qquad t > 0. \tag{3.14}$$

4 Space-time fractional Poisson process Clearly it is possible to combine fractionality in time and space so that we arrive at a space-time fractional Poisson process ${}^\nu N^\alpha(t)$, $t > 0$ of parameters $\alpha, \nu \in (0,1)$. The distribution

$$p_k^{\alpha,\nu}(t) = \Pr\{{}^\nu N^\alpha(t) = k\} \tag{4.1}$$

satisfies the equation

$$\frac{d^\nu}{dt^\nu} p_k^{\alpha,\nu}(t) = -\lambda^\alpha (1 - B)^\alpha p_k^{\alpha,\nu}(t), \qquad k \geq 0, \tag{4.2}$$

and the corresponding probability generating function ${}_\nu G_\alpha(u,t)$ is governed by

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu} {}_\nu G_\alpha(u,t) = -\lambda^\alpha (1-u)^\alpha \, {}_\nu G_\alpha(u,t) \\ {}_\nu G_\alpha(u,0) = 1. \end{cases} \tag{4.3}$$

The solution to (4.3) is given by

$${}_\nu G_\alpha(u,t) = E_{\nu,1}\left(-\lambda^\alpha t^\nu (1-u)^\alpha\right), \qquad |u| \leq 1. \tag{4.4}$$

For $0 < u < 1$ the probability generating function (4.4) has the following representation

$${}_\nu G_\alpha(u,t) = \Pr\left\{\min_{0 \leq k \leq N^\nu(t)} X_k^{\frac{1}{\alpha}} \geq 1 - u\right\}, \qquad 0 < u < 1, \tag{4.5}$$

where the driving process is, in this case, the time-fractional Poisson process dealt with at the beginning of this note. Of course the r.v.'s X_k, $k \geq 1$, are independent and uniformly distributed in $[0,1]$.

5 A further fractional Poisson process Another type of fractional Poisson process can be constructed by considering the process $\widehat{N_\nu}(t)$, $t > 0$, with distribution

$$\Pr\left\{\widehat{N_\nu}(t) = k\right\} = \frac{(\lambda t)^k}{\Gamma(\nu k + 1)} \frac{1}{E_{\nu,1}(\lambda t)}, \qquad k \geq 0. \tag{5.1}$$

and probability generating function

$$G_\nu(u,t) = \frac{E_{\nu,1}(u\lambda t)}{E_{\nu,1}(\lambda t)}, \qquad |u| < 1, t > 0. \tag{5.2}$$

The process with distribution (5.1) does not possesses the fine property of the lack of memory of the Poisson process, has dependent increments and mean value equal to

$$\mathbb{E}\widehat{N}_\nu(t) \;=\; \frac{\lambda t}{\nu}\frac{E_{\nu,\nu}\left(\lambda t\right)}{E_{\nu,1}\left(\lambda t\right)}. \tag{5.3}$$

Fractionality in this case can be justified by observing that (5.2) satisfies the following fractional equation

$$\frac{\partial^\nu}{\partial u^\nu}G_\nu(u,t) \;=\; \lambda t\, G_\nu\left(u^\nu, t\right) \tag{5.4}$$

where the fractional derivative (w.r. to u^ν) must be understood in the sense of Dzerbayshan-Caputo. The distribution (5.1) can be regarded as a weighted Poisson sum as pointed out in Beghin and Macci [3]; Balakrishnan and Kozubowski [1]:

$$\Pr\left\{\widehat{N_\nu}(t) = k\right\} \;=\; \frac{\frac{k!}{\Gamma(\nu k+1)}\Pr\left\{N(t)=k\right\}}{\sum_{j=0}^{\infty}\frac{j!}{\Gamma(\nu j+1)}\Pr\left\{N(t)=j\right\}}, \tag{5.5}$$

with weights $\frac{j!}{\Gamma(\nu j+1)}$, $j \geq 0$. Fractional Poisson processes have been considered in some applications for example by Cahoy [5]; Laskin [8].

6 Fractional pure birth process Some fractional versions of other point processes have been considered. We mention the pure birth process (Orsingher and Polito [10]), the death processes (Orsingher et al. [13]), the Poisson compound process Beghin and Macci [4], the linear birth and death processes (Orsingher and Polito [11]). We just mention here the construction of the non-linear birth process with one primogenitor. The state probabilities

$$p_k^\nu(t) \;=\; \Pr\left\{N^\nu(t) = k | N^\nu(0) = 1\right\} \tag{6.1}$$

are governed by

$$\frac{d^\nu}{dt^\nu}p_k^\nu(t) \;=\; -\lambda_k p_k^\nu(t) + \lambda_{k-1}p_{k-1}^\nu(t), \qquad k \geq 1, \tag{6.2}$$

with

$$p_k^\nu(0) \;=\; \begin{cases} 1, & k = 1, \\ 0, & k > 1, \end{cases} \tag{6.3}$$

where the fractional derivative must be meant as in (2.7). The $\lambda_k > 0$ are the birth rates which become linear when $\lambda_k = k\lambda$. It is shown in Orsingher and Polito [10] that

$$p_k^\nu(t) = \begin{cases} \prod_{j=1}^{k-1}\lambda_j \sum_{m=1}^{k}\frac{E_{\nu,1}(-\lambda_m t^\nu)}{\prod_{l=1, l\neq m}^{m}(\lambda_l - \lambda_m)}, & k > 1, \\ E_{\nu,1}\left(-\lambda_1 t^\nu\right), & k = 1. \end{cases} \tag{6.4}$$

In this linear case the distribution above reduces to

$$p_k^\nu(t) \;=\; \sum_{j=1}^{k}(-1)^{j-1}\binom{k-1}{j-1}E_{\nu,1}\left(-\lambda_j t^\nu\right), \qquad k \geq 1, t > 0, \tag{6.5}$$

and for $\nu = 1$ becomes the geometric distribution of the Yule-Furry model. The mean value of $N^\nu(t)$, $t > 0$, is given by

$$\mathbb{E}\left\{N^\nu(t)|N(0)=1\right\} = E_{\nu,1}\left(\lambda t^\nu\right). \tag{6.6}$$

It is shown that the lesser is the fractionality degree ν then more rapidly the mean (6.6) increases (see Orsingher and Polito [10]). The intertime between successive births has been studied in the linear case in Cahoy and Polito [6] where interesting simulations of this branching process are produced for different values of $\nu \in (0, 1)$. The fractional version of the birth process seems suitable to model rapidly expanding epidemies.

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