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Numerical Solution of Fuzzy Integral Equations

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Abstract

In this paper, a numerical procedure for solving fuzzy Fredholm integral equations of the second kind ($FIEs$) with arbitrary kernels have been investigated and residual minimization method is given and then the proposed algorithm is illustrated with solving some numerical examples.

Keywords: Fuzzy number, Fuzzy linear system, Fuzzy integral equations

1 Introduction

The concept of integration of fuzzy functions was first introduced by Dubois and Prade [5]. The topics of numerical methods for solving fuzzy integral equations have been rapidly growing in recent years and have been studies by authors of [6]. The numerical methods for fuzzy differential equations have been studied by S. Abbasbandy, T. Allahviranloo, [1, 2, 3] and others. Alternative approaches were later suggested by Goetschel and Vaxman [8], Kaleva [10] and others. The structure of this paper is organized as follows:
In section 2, some basic definitions and results on fuzzy numbers, fuzzy integral and the fuzzy linear system is brought. In Section 3, we propose a general method for solving fuzzy Fredholm integral equation of the second kind. In Section 4, we illustrate algorithm by solving some numerical examples. The conclusions are drawn in Section 5.

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2 Preliminaries

Let us now introduce the notation needed in the rest of the paper. We will place a bar over a symbol if it represents a fuzzy number so $\tilde{a}, \tilde{b}, \tilde{c}$ are all fuzzy numbers but a, b, c will denote real numbers. Parametric form of an arbitrary fuzzy number is given in [4] as follows. A fuzzy number $\tilde{u}$ in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r), 0 \leq r \leq 1$, which satisfies the following requirements:

1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$,
2. $\overline{u}(r)$ is a bounded left continuous non-increasing function over $[0, 1]$,
3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$,

The set of all these fuzzy numbers is denoted by E. A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha, \quad 0 \leq r \leq 1$. A popular fuzzy number is the triangular fuzzy number $\tilde{u} = (m, \alpha, \beta)$ which

$$\tilde{u}(x) = \begin{cases} \frac{x-m}{\alpha} + 1, & m - \alpha \leq x \leq m, \\ \frac{m-x}{\beta} + 1, & m \leq x \leq m + \beta, \\ 0, & otherwise. \end{cases}$$

Its parametric form is

$$\underline{u}(r) = m + \alpha(r - 1), \quad \overline{u}(r) = m + \beta(1 - r).$$

By appropriate definitions the fuzzy number space $\{\underline{u}(r), \overline{u}(r)\}$ becomes a convex cone E^1 which is then embedded isomorphically and isometrically into a Banach space.

Definition 1. The $n \times n$ dual linear system

$$\begin{cases} a_{11}\tilde{x}_1 + \cdots + a_{1n}\tilde{x}_n = \tilde{y}_1 + b_{11}\tilde{x}_1 + \cdots + b_{1n}\tilde{x}_n, \\ a_{21}\tilde{x}_1 + \cdots + a_{2n}\tilde{x}_n = \tilde{y}_2 + b_{21}\tilde{x}_1 + \cdots + b_{2n}\tilde{x}_n, \\ \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \\ a_{n1}\tilde{x}_1 + \cdots + a_{nn}\tilde{x}_n = \tilde{y}_n + b_{n1}\tilde{x}_1 + \cdots + b_{nn}\tilde{x}_n, \end{cases} \tag{1}$$

where the coefficient matrix $A = (a_{ij})$ and $B = (b_{ij}), 1 \leq i, j \leq n$ is a crisp $n \times n$ matrix, $\tilde{x}^t = (\tilde{x}_1, \ldots, \tilde{x}_n)$ be a $n \times 1$ vector of fuzzy numbers $\tilde{x}_j$ and $\tilde{y}^t = (\tilde{y}_1, \ldots, \tilde{y}_n)$ be a $n \times 1$ vector of fuzzy numbers $\tilde{y}_i$ is called a dual fuzzy linear system (DFLS).

For arbitrary fuzzy numbers $\tilde{x} = (\underline{x}(r), \overline{x}(r)), \tilde{y} = (\underline{y}(r), \overline{y}(r))$ and real number k, we may define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as

(a) $\tilde{x} = \tilde{y}$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$,

(b) $\tilde{x} + \tilde{y} = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$,

(c) $k\tilde{x} = \begin{cases} (k\underline{x}, k\overline{x}), & k \geq 0, \\ (k\overline{x}, k\underline{x}), & k < 0. \end{cases}$

Definition 2. A fuzzy number vector $(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n)^t$ given by

$$\tilde{x}_i = (\underline{x}_i(r), \overline{x}_i(r)), \quad 1 \leq i \leq n, 0 \leq r \leq 1,$$

is called a solution of the fuzzy linear system (1) if

$$\begin{cases} \underline{\sum_{j=1}^n a_{ij}x_j} = \sum_{j=1}^n \underline{a_{ij}x_j} = \underline{y}_i = \underline{\sum_{j=1}^n b_{ij}x_j} = \sum_{j=1}^n \underline{b_{ij}x_j}, \\ \overline{\sum_{j=1}^n a_{ij}x_j} = \sum_{j=1}^n \overline{a_{ij}x_j} = \overline{y}_i = \overline{\sum_{j=1}^n b_{ij}x_j} = \sum_{j=1}^n \overline{b_{ij}x_j}. \end{cases}$$

If, for a particular $i, a_{ij} > 0$ and $b_{ij} > 0, 1 \leq j \leq n$, we simply get

$$\sum_{j=1}^n a_{ij}\underline{x}_j = \underline{y}_i + \sum_{j=1}^n b_{ij}\underline{x}_j, \qquad \sum_{j=1}^n a_{ij}\overline{x}_j = \overline{y}_i + \sum_{j=1}^n b_{ij}\overline{x}_j.$$

The following theorem guarantees the existence of a fuzzy solution for general case. Consider the dual fuzzy linear system (6), and transform its $n \times n$ coefficient matrix A and B in to $(2n) \times (2n)$ matrices as:

$$\begin{cases} s_{11}\underline{x}_1 + \cdots + s_{1n}\underline{x}_n + s_{1,n+1}(-\overline{x}_1) + \cdots + s_{1,2n}(-\overline{x}_n) = \underline{y}_1 + \\ \qquad t_{11}\underline{x}_1 + \cdots + t_{1n}\underline{x}_n + t_{1,n+1}(-\overline{x}_1) + \cdots + t_{1,2n}(-\overline{x}_n), \\ \vdots \\ s_{n1}\underline{x}_1 + \cdots + s_{nn}\underline{x}_n + s_{n,n+1}(-\overline{x}_1) + \cdots + s_{n,2n}(-\overline{x}_n) = \underline{y}_n + \\ \qquad t_{n1}\underline{x}_1 + \cdots + t_{nn}\underline{x}_n + t_{n,n+1}(-\overline{x}_1) + \cdots + t_{n,2n}(-\overline{x}_n), \\ s_{n+1,1}\underline{x}_1 + \cdots + s_{n+1,n}\underline{x}_n + s_{n+1,n+1}(-\overline{x}_1) + \cdots + s_{n+1,2n}(-\overline{x}_n) = -\overline{y}_1 + \\ \qquad t_{n+1,1}\underline{x}_1 + \cdots + t_{n+1,n}\underline{x}_n + t_{n+1,n+1}(-\overline{x}_1) + \cdots + t_{n+1,2n}(-\overline{x}_n), \\ \vdots \\ s_{2n,1}\underline{x}_1 + \cdots + s_{2n,n}\underline{x}_n + s_{2n,n+1}(-\overline{x}_1) + \cdots + s_{2n,2n}(-\overline{x}_n) = -\overline{y}_n + \\ \qquad t_{2n,1}\underline{x}_1 + \cdots + t_{2n,n}\underline{x}_n + t_{2n,n+1}(-\overline{x}_1) + \cdots + t_{2n,2n}(-\overline{x}_n), \end{cases}$$

where s_{ij} and t_{ij} are determined as follows:

$$\begin{aligned} &a_{ij} \geq 0 \Longrightarrow s_{ij} = a_{ij},\ s_{i+n,j+n} = a_{ij}, \\ &a_{ij} < 0 \Longrightarrow s_{i,j+n} = -a_{ij},\ s_{i+n,j} = -a_{ij}, \\ &b_{ij} \geq 0 \Longrightarrow t_{ij} = b_{ij},\ t_{i+n,j+n} = b_{ij}, \\ &b_{ij} < 0 \Longrightarrow t_{i,j+n} = -b_{ij},\ t_{i+n,j} = -b_{ij}, \end{aligned} \tag{2}$$

and any s_{ij} and t_{ij} which is not determined by (2) is zero. Using matrix notation we get

$$SX = Y + TX, \tag{3}$$

therefore, we have:

$$(S - T)X = Y, \tag{4}$$

where $S = (s_{ij}) \geq 0$ and $T = (t_{ij}) \geq 0, 1 \leq i, j \leq 2n$, and

$$X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix}, \quad Y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ -\overline{y}_1 \\ \vdots \\ -\overline{y}_n \end{bmatrix}. \tag{5}$$

For example:
Consider the dual fuzzy linear system

$$\begin{cases} \tilde{x}_1 - \tilde{x}_2 = \tilde{y}_1 + 2\tilde{x}_1 + \tilde{x}_2, \\ \tilde{x}_1 + 2\tilde{x}_2 = \tilde{y}_2 + \tilde{x}_1 - 2\tilde{x}_2. \end{cases} \tag{6}$$

Let $\underline{y}_1 = r, \overline{y}_1 = 2 - r$ and $\underline{y}_2 = 4 + r, \overline{y}_2 = 7 - 2r$, the extended 4×4 matrices are

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad T = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix},$$

and

$$Y = \begin{bmatrix} r \\ 4 + r \\ r - 2 \\ 2r - 7 \end{bmatrix}, \quad X = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\overline{x}_1 \\ -\overline{x}_2 \end{bmatrix}.$$

We obtain that the system (6) is equivalent to the function equation system

$$SX = Y + TY.$$

Consequently,

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\overline{x}_1 \\ -\overline{x}_2 \end{bmatrix} = \begin{bmatrix} r \\ 4 + r \\ r - 2 \\ 2r - 7 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\overline{x}_1 \\ -\overline{x}_2 \end{bmatrix}.$$

Also,

$$(S - T)X = Y.$$

The structure of S and T implies that:

$$S = \begin{pmatrix} C & D \\ D & C \end{pmatrix}, \quad T = \begin{pmatrix} E & F \\ F & E \end{pmatrix},$$

where C and E contains the positive entries of A and B respectively, and D and F the absolute values of the negative entries of A and B, i.e. $A = C - D$ and $B = E - F$. Therefore

$$S - T = \begin{pmatrix} C - E & D - F \\ D - F & C - E \end{pmatrix},$$

Theorem 1. The matrix $S - T$ is nonsingular if and if the matrix $(C + D) - (E + F)$ and $(C + F) - (E + D)$ are both nonsingular.
Proof. The same as the proof of Theorem 1 in [7].
Assuming that $S - T$ is nonsingular we obtain of the Eq. (4)

$$X = (S - T)^{-1} Y \tag{7}$$

Theorem 2. If $(S - T)^{-1}$exists it must have the same structure as S, i.e.

$$(S - T)^{-1} = \begin{pmatrix} G & H \\ H & G \end{pmatrix},$$

and

$$G = \frac{1}{2}\left[\ ((C + D) - (E + F))^{-1} + ((C + F) - (E + D))^{-1}\ \right],$$

$$H = \frac{1}{2}\left[\ ((C + D) - (E + F))^{-1} - ((C + F) - (E + D))^{-1}\ \right],$$

Proof. see [7].
Theorem 3. The unique solution X of Eq. (7) is a fuzzy vector for arbitrary Y if and only if $(S - T)^{-1}$ is nonnegative, i.e.

$$((S - T)^{-1})_{ij} \geq 0, \qquad 1 \leq i \leq 2n, \qquad 1 \leq j \leq 2n.$$

Proof. The same as the proof of Theorem 3 in [7].
Definition 3. Let $X = \{(\underline{x}_i(r), \overline{x}_i(r)), 1 \leq i \leq n\}$ denotes the unique solution of (3), if $\underline{y}_i(r), \overline{y}_i(r)$ are linear functions of r, then the fuzzy number vector $U = \{(\underline{u}_i(r), \overline{u}_i(r)), 1 \leq i \leq n\}$ defined by

$$\underline{u}_i(r) = min\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1)\},$$
$$\overline{u}_i(r) = max\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1)\}.$$

is called the fuzzy solution of (3). If $(\underline{x}_i(r), \overline{x}_i(r)), 1 \leq i \leq n$, are all fuzzy numbers then $\underline{u}_i(r) = \underline{x}_i(r), \overline{u}_i(r) = \overline{x}_i(r)$, and then U is called a strong fuzzy solution. Otherwise, U is a weak fuzzy solution.

We will next define a metric D in E^1 and the fuzzy function notation [8].

Definition 4. For arbitrary fuzzy numbers $\tilde{u} = (\underline{u}, \overline{u})$ and $\tilde{v} = (\underline{v}, \overline{v})$ the quantity

$$D(\tilde{u}, \tilde{v}) = sup_{0 \leq r \leq 1}\{max[|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|]\} \tag{8}$$

is the distance between $\tilde{u}$ and $\tilde{v}$. It is shown [11] that (E^1, D) is a complete metric space.

Definition 5. Let $\tilde{U} = (\tilde{u_1}, \ldots, \tilde{u_n})$ and $\tilde{V} = (\tilde{v_1}, \ldots, \tilde{v_n})$, $\dot{D}(\tilde{U}, \tilde{V})$ is:

$$\dot{D}(\tilde{U}, \tilde{V}) = \begin{bmatrix} D(\tilde{u}_1, \tilde{v}_1) \\ \vdots \\ D(\tilde{u}_n, \tilde{v}_n) \end{bmatrix} \tag{9}$$

Definition 6. A function $\tilde{f} : \mathbb{R}^1 \longrightarrow E^1$ is called a fuzzy function. If for arbitrary fixed $t_0 \in \mathbb{R}^1$ and $\varepsilon > 0$, a $\delta > 0$ such that

$$| t - t_0 | < \delta \Longrightarrow D[\tilde{f}(t), \tilde{f}(t_0)] < \epsilon \tag{10}$$

exist, f is said to be continuous.

3 Fuzzy integral equations

The Fredholm integral equation of the second kind is

$$x(s) = f(s) + \lambda \int_a^b k(s,t)x(t)dt \tag{11}$$

where $\lambda > 0$, $k(s,t)$ is an arbitrary kernel function over the square $a \leq s, t \leq b$ and $f(t)$ is a function of $t : a \leq t \leq b$, [9]. If $f(t)$ is a crisp function then the solution of Eqs. (11) is crisp as well. However, if $f(t)$ is a fuzzy function this equation may only possess fuzzy solution. Therefore, we have

$$\tilde{x}(s) = \tilde{f}(s) + \lambda \int_a^b k(s,t)\tilde{x}(t)dt. \tag{12}$$

Sufficient conditions for the existence of a unique solution to the fuzzy Fredholm integral equation of the second kind, i.e. to Eq. (12) where $\tilde{f}(t)$ is a fuzzy function, are given in [12].

We consider now the numerical solution of fuzzy Fredholm integral equations of the second kind Eq. (12), which we write in the form:

$$\tilde{x} = \tilde{f} + \lambda K \tilde{x}. \tag{13}$$

The exact solution of integral equation Eq. (12) is:

$$\tilde{x}(s) = \sum_{i=1}^{\infty} \tilde{a}_i h_i(s) \tag{14}$$

in truncated form

$$\tilde{x}(s) \approx \tilde{x}_n(s) = \sum_{i=1}^{n} \tilde{a}_i h_i(s), \tag{15}$$

where the set $\{h_i\}$ is complete and orthogonal in $\ell^2(a,b)$ (see [**?**]). For finding approximation solution we must indicate coefficients $\tilde{a}_i$.

From Eq. (15) we obtain

$$\sum_{j=1}^{n} \tilde{a}_j h_j(s) = \tilde{f}(s) + \lambda \sum_{j=1}^{n} \tilde{a}_j \int_a^b k(s,t) h_j(t) dt. \tag{16}$$

We have n unknown parameters in the form $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n$ which for finding them, we need to n equation, so by using n point $s_1, s_2, \ldots, s_n$ in interval $[a,b]$:

$$\sum_{j=1}^{n} h_j(s_i) \tilde{a}_j = \tilde{f}(s_i) + \lambda \sum_{j=1}^{n} \int_a^b k(s_i,t) h_j(t) dt \tilde{a}_j \qquad i = 1, \cdots, n \tag{17}$$

therefore we have:

$$A\tilde{a} = \tilde{f} + B\tilde{a}, \tag{18}$$

where the coefficients matrix $A = (a_{ij}), 1 \leq i,j \leq n$, and $B = (b_{ij}), 1 \leq i,j \leq n$, are crisp and $\tilde{f} = (\tilde{f}_i), 1 \leq i \leq n$, is an arbitrary fuzzy number vector, where

$$a_{ij} = h_j(s_i), \qquad b_{ij} = \lambda \int_a^b k(s_i,t) h_j(t) dt, \qquad i,j = 1, \cdots, n$$

3.1 Residual minimization method

The simplest method conceptually again appeals to approximation theory. We write the integral equation in the form (again we set $\lambda = 1$)

$$L\tilde{x} = \tilde{f}, \quad L = I - K, \tag{19}$$

and introduce the residual function r_n and error function ε_n

$$r_n = \dot{D}(\tilde{f}, L\tilde{x}_n), \tag{20}$$
$$\varepsilon_n = \dot{D}(\tilde{x}, \tilde{x}_n). \tag{21}$$

To compute r_n requires no knowledge of $\tilde{x}$ but, since $\dot{D}(\tilde{f}, L\tilde{x}) = 0$, we have the identity:

$$r_n = \dot{D}(\tilde{f}, L\tilde{x}_n) - \dot{D}(\tilde{f}, L\tilde{x}) = L\dot{D}(\tilde{x}_n, \tilde{x}) = L\varepsilon_n. \tag{22}$$

From (19) and (22) we have at once

$$\| r_n \| \leq (1+ \| K \|) \| \varepsilon_n \| . \tag{23}$$

That is,

$$\| \varepsilon_n \| \geq \frac{\| r_n \|}{1+ \| K \|}. \tag{24}$$

Thus a small residual is a necessary condition for a small error. We would rather have an upper bound on ε_n of course; this is harder to provide in general and we content ourselves for now with the following. We rewrite (22) as

$$\varepsilon_n = r_n + K\varepsilon_n$$

whence

$$\| \varepsilon_n \| \leq \| r_n \| + \| K \| . \| \varepsilon_n \| \qquad and \;\; hence \;\; if \quad \| K \| < 1 \tag{25}$$
$$\| \varepsilon_n \| \leq \frac{\| r_n \|}{1- \| K \|}. \tag{26}$$

4 Numerical examples

Example 1. Consider the fuzzy integral equation (12) where

$$\underline{f}(s; r) = s^3(r^2 + r),$$
$$\overline{f}(s; r) = s^3(4 - r^3 - r),$$

and kernel

$$k(s, t) = s + 1, \qquad -1 \leq s, t \leq 1.$$

and $a=-1, b=1$. The exact solution in this case is given by

$$\underline{x}(s;r) = s^3(r^2+r),$$
$$\overline{x}(s;r) = s^3(4-r^3-r).$$

Let

$$h_1(s)=1, h_2(s)=s^3,$$

and

$$s_1=-1, s_2=1.$$

From the Eqs. (17) and (18):

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{bmatrix} = \begin{bmatrix} \tilde{f}(s_1) \\ \tilde{f}(s_2) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{bmatrix},$$

the Extended 4×4 matrices are

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

and the solution of Eq. (7) is:

$$\begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ -\overline{a}_1 \\ -\overline{a}_2 \end{bmatrix} = (S-T)^{-1}F = \begin{bmatrix} -\frac{r^3}{4} - \frac{r^2}{4} - \frac{r}{2} + 1 \\ -\frac{3}{4}r^3 + \frac{r^2}{4} - \frac{r}{2} + 3 \\ -\frac{r^3}{4} - \frac{r^2}{4} - \frac{r}{2} + 1 \\ \frac{r^3}{4} - \frac{3}{4}r^2 - \frac{r}{2} - 1 \end{bmatrix}. \tag{27}$$

The fact that $\tilde{a}_1$ and $\tilde{a}_2$ are not fuzzy numbers. Therefore the fuzzy solution of the Eq. (27) is a weak fuzzy solution. The fuzzy approximate solution in this case is given by

$$\tilde{x}(s) \approx \tilde{x}_2(s) = \sum_{i=1}^{2} \tilde{a}_i h_i(s)$$

where

$$\tilde{a}_1 = (\tfrac{r^3}{4} + \tfrac{r^2}{4} + \tfrac{r}{2} - 1, -\tfrac{r^3}{4} - \tfrac{r^2}{4} - \tfrac{r}{2} + 1)$$
$$\tilde{a}_2 = (-\tfrac{r^3}{4} + \tfrac{3}{4}r^2 + \tfrac{r}{2} + 1, -\tfrac{3}{4}r^3 + \tfrac{r^2}{4} - \tfrac{r}{2} + 3).$$

Example 2. Consider the fuzzy integral equation (12) where

$$\underline{f}(s;r) = (\frac{s}{2} - \frac{1}{3})r,$$
$$\overline{f}(s;r) = (\frac{s}{2} - \frac{1}{3})(2-r),$$

and kernel

$$k(s,t) = s + t, \qquad 0 \leq s, t \leq 1.$$

and $a = 0, b = 1$. The exact solution in this case is given by

$$\underline{x}(s;r) = sr,$$
$$\overline{x}(s;r) = s(2-r).$$

Let

$$h_1(s) = s, h_2(s) = s^3, h_3(s) = s^5,$$

and

$$s_1 = 0, s_2 = \frac{1}{2}, s_3 = 1.$$

From the Eqs. (17) and (18):

$$\begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{8} & \frac{1}{32} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{bmatrix} = \begin{bmatrix} \tilde{f}(s_1) \\ \tilde{f}(s_2) \\ \tilde{f}(s_3) \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & \frac{1}{5} & \frac{1}{7} \\ \frac{7}{12} & \frac{13}{40} & \frac{19}{84} \\ \frac{5}{6} & \frac{9}{20} & \frac{13}{42} \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{bmatrix},$$

the Extended 6×6 matrices are

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{8} & \frac{1}{32} & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{8} & \frac{1}{32} \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & 0 & 0 & 0 \\ \frac{7}{12} & \frac{13}{40} & \frac{19}{84} & 0 & 0 & 0 \\ \frac{5}{6} & \frac{9}{20} & \frac{13}{42} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{5} & \frac{1}{7} \\ 0 & 0 & 0 & \frac{7}{12} & \frac{13}{40} & \frac{19}{84} \\ 0 & 0 & 0 & \frac{5}{6} & \frac{9}{20} & \frac{13}{42} \end{bmatrix}$$

and the solution of Eq. (7) is:

$$\begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \\ -\overline{a}_1 \\ -\overline{a}_2 \\ -\overline{a}_3 \end{bmatrix} = (S-T)^{-1}F = \begin{bmatrix} r \\ 0 \\ 0 \\ -2+r \\ 0 \\ 0 \end{bmatrix}, \tag{28}$$

i.e.

$$\begin{aligned} \tilde{a}_1 &= (r, 2-r) \\ \tilde{a}_2 &= (0,0) \\ \tilde{a}_3 &= (0,0). \end{aligned}$$

Here $\underline{a}_1 \leq \overline{a}_1, \underline{a}_2 \leq \overline{a}_2, \underline{a}_3 \leq \overline{a}_3$ are monotonic decreasing functions. Therefore the fuzzy solution of the Eq. (28) is a strong fuzzy solution. The fuzzy approximate solution in this case is a approximate solution given by

$$\tilde{x}(s) \approx \tilde{x}_3(s) = \sum_{i=1}^{3} \tilde{a}_i h_i(s)$$

Example 3. Consider the fuzzy integral equation (12) where

$$\begin{aligned} \underline{f}(s;r) &= -\frac{2}{\pi}\cos s(r^2+r), \\ \overline{f}(s;r) &= -\frac{2}{\pi}\cos s(3-r), \end{aligned}$$

and kernel

$$k(s,t) = \cos(s-t), \qquad \lambda = \frac{4}{\pi},$$

and $a = 0, b = \frac{\pi}{2}$. The exact solution in this case is given by

$$\begin{aligned} \underline{x}(s;r) &= \sin s(r^2+r) = (s - \frac{s^3}{3!} + \frac{s^5}{5!} - \dots)(r^2+r), \\ \overline{x}(s;r) &= \sin s(3-r) = (s - \frac{s^3}{3!} + \frac{s^5}{5!} - \dots)(3-r). \end{aligned}$$

Let

$$h_1(s) = s, h_2(s) = s^3, h_3(s) = s^5,$$

and

$$s_1 = 0, s_2 = \frac{\pi}{4}, s_3 = \frac{\pi}{2} - 0.01.$$

From the Eqs. (17) and (18):

$$\begin{bmatrix} 0 & 0 & 0 \\ \frac{\pi}{4} & (\frac{\pi}{4})^3 & (\frac{\pi}{4})^5 \\ \frac{\pi}{4}-0.01 & (\frac{\pi}{4}-0.01)^3 & (\frac{\pi}{4}-0.01)^5 \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{bmatrix} = \begin{bmatrix} \tilde{f}(s_1) \\ \tilde{f}(s_2) \\ \tilde{f}(s_3) \end{bmatrix} +$$

$$\begin{bmatrix} 0.726765132 & 0.574239947 & 0.6913436 \\ 1.4142126 & 1.668453 & 2.6463 \\ 1.2805 & 1.791 & 3.0578 \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{bmatrix},$$

the Extended 6×6 matrices are

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\pi}{4} & (\frac{\pi}{4})^3 & (\frac{\pi}{4})^5 & 0 & 0 & 0 \\ \frac{\pi}{4}-0.01 & (\frac{\pi}{4}-0.01)^3 & (\frac{\pi}{4}-0.01)^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\pi}{4} & (\frac{\pi}{4})^3 & (\frac{\pi}{4})^5 \\ 0 & 0 & 0 & \frac{\pi}{4}-0.01 & (\frac{\pi}{4}-0.01)^3 & (\frac{\pi}{4}-0.01)^5 \\ 0 & 0 & 0 & \frac{\pi}{4}-0.01 & (\frac{\pi}{4}-0.01)^3 & (\frac{\pi}{4}-0.01)^5 \end{bmatrix},$$

$$T = \begin{bmatrix} 0.726765132 & 0.574239947 & 0.6913436 & 0 & 0 & 0 \\ 1.4142126 & 1.668453 & 2.6463 & 0 & 0 & 0 \\ 1.2805 & 1.791 & 3.0578 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.726765132 & 0.574239947 & 0.6913436 \\ 0 & 0 & 0 & 1.4142126 & 1.668453 & 2.6463 \\ 0 & 0 & 0 & 1.2805 & 1.791 & 3.0578 \end{bmatrix}$$

and the solution of Eq. (7) is:

$$\begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \underline{a}_3 \\ -\overline{a}_1 \\ -\overline{a}_2 \\ -\overline{a}_3 \end{bmatrix} = (S-T)^{-1}F = \begin{bmatrix} 1.0112r^2 + 1.012r \\ -0.196r^2 - 0.196r \\ 0.0192r^2 + 0.0192r \\ 1.012r - 3.036 \\ 0.587 - 0.196r \\ 0.00481r - 0.057 \end{bmatrix}. \tag{29}$$

i.e.

$$\begin{aligned} \tilde{a}_1 &= (1.0112r^2 + 1.012r, -1.012r + 3.036) \\ \tilde{a}_2 &= (-0.196r^2 - 0.196r, -0.587 + 0.196r) \\ \tilde{a}_3 &= (0.0192r^2 + 0.0192r, -0.00481r + 0.057). \end{aligned}$$

The fact that $\tilde{a}_2$ is not a fuzzy number. Therefore the fuzzy solution of the Eq. (29) is a weak fuzzy solution. The fuzzy approximate solution in this case is a approximate solution given by

$$\tilde{x}(s) \approx \tilde{x}_3(s) = \sum_{i=1}^{3} \tilde{\tilde{a}}_i h_i(s)$$

where

$$\tilde{a}_1 = (1.0112r^2 + 1.012r, -1.012r + 3.036)$$
$$\tilde{a}_2 = (-0.587 + 0.196r, -0.196r^2 - 0.196r)$$
$$\tilde{a}_3 = (0.0192r^2 + 0.0192r, -0.00481r + 0.057).$$

5 Conclusions

In this paper, we proposed a numerical method for solving fuzzy Fredholm integral equation of the second kind. The resulted approximate solutions from expansion method may be a strong or weak fuzzy solutions.

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