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On α-topological vector space

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Abstract
The purpose of this article is to continue the study of α-topological vector space in the spirit of α-open set, α-closed set, α-continuous mapping from the previous paper by O. Njastad [10], A.S. Mashhour [9], S.N. Maheshwari and S.S. Thakur [7, 8]. Several properties of α-topological vector space are obtained. There are many more properties of the type translation, dilation and deep structure of the space are described in detail.

Keywords
α-open sets, α-closed sets, α-topological vector space.

AMS Subject Classification
57N17, 57N99.

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Contents

1. Introduction and Preliminaries

Ever since the advent of topological vector spaces given by Kolmogroff [6] in 1934, these spaces remain a center of attraction for many researchers and mathematicians because of their vast scope and existing properties. Hyers [2], Wenhausen [18], Schaefer [15] and many others have contributed a lot in the field of topological vector spaces. In this day and age, topological vector spaces are important and fundamental notion in fixed point theory, operator theory, variational inequalities, vector equilibrium problems and many other advanced branches of mathematics.

We recall some generalizations and similar structures of topological vector spaces added to their developments: strongly preirresolute topological vector space [11], s-topological vector space [3], irresolute topological vector space [5], almost pretopological vector spaces [16], almost s-topological vector spaces [14] and β-topological vector spaces [17].

In our paper, we introduce the idea of α-topological vector spaces and generalize the properties.

Definition 1.1. *A subset A of a topological space X is called α-open [7] if $A \subseteq Int(Cl(Int(A)))$.*

It is clear from the definition that every open set is α-open but not conversely. We will present an example in this regard.

Example 1.2. *Let X be a space of real numbers with the usual topology induced on it. Consider $F = \{\frac{1}{n} : n \in \mathbb{N}\}$, where $\mathbb{N}$ being the set of natural numbers. Then F^c is an α-open set which is not open.*

A set A in a topological space X is α-closed if its complement is α-open or equivalently, $Cl(Int(Cl(A))) \subseteq A$. The family of all α-open(resp. α-closed) sets in X will be denoted by αO(X)(resp. αC(X)). If A is any subset of topological space X, then
(1) the intersection of all α-closed set in X containing A is called α-closure of A [7] and is denoted by αCl(A), and
(2) the union of all α-open sets in X that are contained in A is called α-interior [7] of A and is denoted by αInt(A).
We will make use of the notation $\mathscr{N}_0(X)$ and $N_0(X)$ to denote the open neighborhood and α-open neighborhood of origin in a topological space X respectively.

Definition 1.3. *A topological space is said to be α-compact [7]if every cover of X by α-open sets has a finite subcover.*

2. α-Topological Vector Spaces

Let us introduce some notations. By $\mathbb{K}$, we mean the field of reals or complex numbers, with its usual topology. Scalars are the elements of $\mathbb{K}$. In this section, we define α-topological vector spaces and present some examples of them. We study some basic properties of α-topological vector spaces.

Definition 2.1. *Let E be a vector space over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with the standard topology. Let τ be a topology on E such that the following conditions are satisfied:*
(1) For each $x, y \in E$ and each open set $W \subseteq E$ containing $x+y$, there exist α-open sets U and V in E containing x and y respectively, such that $U+V \subseteq W$, and
(2) For each $\lambda \in \mathbb{K}$, $x \in E$ and each open set $W \subseteq E$ containing λx, there exist α-open sets U and V in E containing x and y respectively, such that $U.V \subseteq W$.
Then the pair $(E_{(\mathbb{K})}, \tau)$ is called an α-topological vector space (written in short, αTVS).

Given below is an example of α-topological vector spaces.

Example 2.2. *Let $E = \mathbb{R}$ be the real vector space over the field $\mathbb{K}$ endowed with the standard topology τ. Then $E_{(\mathbb{K})}, \tau)$ is an α-topological vector space.*

Using the fact that every open set is α-open, it follows that every topological vector space is an α-topological vector space but the converse is not true in general.

From here on, we simply write E for an α-topological vector space $(E_{(\mathbb{K})}, \tau)$. Let us now discuss some basic properties of α-topological vector spaces.

Theorem 2.3. *For any open subset A of an α-topological vector space E, the following are true:*
(a) $x+A \in \alpha O(E)$ for each $x \in E$.
(b) $\lambda A \in \alpha O(E)$ for each non-zero scalar λ.

Proof. (a) Let $y \in x+A$. Since E is α-topological vector space, there exist α-open sets $U, V \in \alpha O(E)$ containing $-x$ and y respectively, such that $U+V \subseteq A$. In particular, $-x+V \subseteq U+V \subseteq A \Rightarrow V \subseteq x+A \Rightarrow y \in \alpha Int(x+A)$ and hence $x+A = \alpha Int(x+A)$. This proves that $x+A$ is α-open set in E.

(b) Let $x \in \lambda A$ be an arbitrary. By definition of α-topological vector spaces, there exist α-open set U in $\mathbb{K}$ containing $\frac{1}{\lambda}$ and V in E containing x such that $U.V \subseteq A \Rightarrow x \in V \subseteq \lambda A \Rightarrow x \in \alpha Int(\lambda A) \Rightarrow \lambda A = \alpha Int(\lambda A)$. Thus $\lambda A \in \alpha O(E)$. □

Theorem 2.4. *For any open subset A of an α-topological vector space E, the following are true:*
(a) $x+A \subseteq Int(Cl(Int(x+A)))$ for each $x \in E$.
(b) $\lambda A \subseteq Int(Cl(Int(\lambda A)))$ for each non-zero scalar λ.

Proof. Trivially from Theorem 2.3. □

Theorem 2.5. *Let F be any closed subset of an α-topological vector space E. Then the following are true:*
(a) $x+F \in \alpha C(E)$ for each $x \in E$.
(b) $\lambda F \in \alpha C(E)$ for each non-zero scalar λ.

Proof. (a) Suppose that $y \in \alpha Cl(x+F)$. Consider $z = -x+y$ and let W be any open set in E such that $z \in W$. Then there exist α-open sets U and V in E such that $-x \in U, y \in V$ and $U+V \subseteq W$. Since $y \in \alpha Cl(x+F), (x+F) \cap V \neq \emptyset$. So, there is $a \in (x+F) \cap V$. Now, $-x+a \in F \cap (U+V) \subseteq F \cap W \Rightarrow F \cap W \neq \emptyset \Rightarrow z \in Cl(F) = F \Rightarrow y \in x+F$ and hence $x+F = \alpha Cl(x+F)$. This proves that $x+F$ is α-closed set in E.

(b) Assume that $x \in \alpha Cl(\lambda A)$. Consider W be open neighborhood of $y = \frac{1}{\lambda}x$ in E. Since E is α-TVS, there exist α-open sets U in $\mathbb{K}$ containing $\frac{1}{\lambda}$ and V in E containing x such that $U.V \subseteq W$. By hypothesis, we have $(\lambda F) \cap V \neq \emptyset$. Therefore, there is $a \in (\lambda F) \cap V$. Now $\frac{1}{\lambda}a \in F \cap (U.V) \subseteq F \cap W \Rightarrow F \cap W \neq \emptyset \Rightarrow y \in Cl(F) = F \Rightarrow x \in \lambda F$ and thereby $\lambda F = \alpha Cl(\lambda F)$. Hence $\lambda F \in \alpha C(E)$. □

An immediate consequence of Theorem 2.5 follows in the next theorem:

Theorem 2.6. *For any closed subset F of an α-topological vector space E, the following are true:*
(a) $Cl(Int(Cl(x+F))) \subseteq x+F$ for each $x \in E$.
(b) $Cl(Int(Cl(\lambda F))) \subseteq \lambda F$ for each non-zero scalar λ.

A simple result that follows from Theorem 2.3 and Theorem 2.5 is the following.

Theorem 2.7. *Let A be any subset of an α-topological vector space E. Then, for each $x \in E$, the following assertion holds:*
(a) $Cl(Int(Cl(x+A))) \subseteq x+Cl(A)$.
(b) $x+Cl(Int(Cl(A))) \subseteq Cl(x+A)$.
(c) $x+Int(A) \subseteq Int(Cl(Int(x+A)))$.
(d) $Int(x+A) \subseteq x+Int(Cl(Int(A)))$.

Theorem 2.8. *Let E be an α-topological vector space. Then for $A \subseteq E$ and $0 \neq \lambda \in \mathbb{K}$, we have*
(a) $Cl(Int(Cl(\lambda A))) \subseteq \lambda Cl(A)$.
(b) $\lambda Cl(Int(Cl(A))) \subseteq Cl(\lambda A)$.

Proof. Straightforward □

Taking notice to Theorem 2.7(a) and Theorem 2.8(a), we find that the former is a generalization of Theorem 2.6(a) while the latter is an improvement of Theorem 2.6(b).

Theorem 2.9. *Let A be any open set in an α-topological vector space E. Then $Cl(x+A) = x+Cl(A)$ for each $x \in E$.*

Proof. Since A is open, we have $A \subseteq Int(Cl(A)) \subseteq Cl(A) \Rightarrow Cl(A) \subseteq Cl(Int(Cl(A)))$. In view of Theorem 2.7(b), $x+Cl(Int(Cl(A))) \subseteq Cl(x+A)$ and hence $x+Cl(A) \subseteq Cl(x+A)$.
Since A is open, by Theorem 2.3, $x+A$ is α-open set in E and thus, $x+A \subseteq Int(Cl(Int(x+A))) \subseteq Int(Cl(x+A))$. This implies that $Cl(x+A) \subseteq Cl(Int(Cl(x+A)))$. By theorem 2.7(a), we obtain $Cl(x+A) \subseteq x+Cl(A)$. Collecting the facts, we get the assertion. □

Theorem 2.10. *Let A be any open set in an α-topological vector space E. Then $Cl(\lambda A) = \lambda Cl(A)$ for each non-zero scalar λ.*

Proof. By theorem 2.8, $Cl(Int(Cl(\lambda A))) \subseteq \lambda Cl(A)$. Since A is open, by theorem 2.3, λA is α-open in E and hence $Cl(\lambda A) = Cl(Int(Cl(\lambda A)))$. Thus, $Cl(\lambda A) \subseteq \lambda Cl(A)$.

Next, since A is open, $Cl(A) \subseteq Cl(Int(Cl(A)))$. On utilizing theorem 2.8, we get $\lambda Cl(Int(Cl(A))) \subseteq Cl(\lambda A)$ and thereby it follows that $\lambda Cl(A) \subseteq Cl(\lambda A)$. Hence $Cl(\lambda A) = \lambda Cl(A)$. So, the proof is finished. □

Definition 2.11. *A mapping $f : X \longrightarrow Y$ from a topological space X to a topological space Y is called α-continuous [9] if for each $x \in X$ and each open set V in Y containing f(x), there exist an α-open set U in X containing x such that $f(U) \subseteq V$.*

Definition 2.12. *Let E be an α-topological vector space. Then a subset $A \subseteq E$ is called*
(a) symmetric if for each $x \in A$, $-x \in A$; that is, $A = -A$.
(b) absorbing if for each $x \in E$, there exist some scalar $\rho > 0$ such that $\lambda x \in A$ for all $\lambda \in \mathbb{K}$ with $|\lambda| < \rho$.
(c) balanced if for every $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1, \lambda A \subseteq A$.
(d) convex if for every $x, y \in A, \lambda x + (1-\lambda)y \in A, \forall \lambda$ with $0 \leq \lambda \leq 1$.

If E is an α-topological vector space. Then for every $V \in \mathscr{N}_0$, there exist $V_1, V_2 \in N_0(E)$ such that $V_1 + V_2 \subseteq V$. Set $U = V_1 \cap V_2 \cap (-V_1) \cap (-V_2)$.

On considering theorem 2.3 and using the fact that the family of all α-open sets forms a topology on E, we find that the $U \in N_0(E)$ such that $U = -U$. Hence we have:

Theorem 2.13. *Let E be an α-topological vector space. Then, for every $V \in \mathscr{N}_0$, there exist a symmetric $U \in N_0(E)$ such that $U + U \subseteq V$.*

Corollary 2.14. *Let E be an α-topological vector space. Then, for every $V \in \mathscr{N}_x$, there exist a symmetric $U \in N_0(E)$ such that $x + U + U \subseteq V$.*

Theorem 2.15. *Let E be an α-topological vector space. Then*
(1) Every convex $V \in \mathscr{N}_0$ is absorbing.
(2) For every convex $V \in \mathscr{N}_0$, there exist a balanced $U \in N_0(X)$ such that $U \subseteq V$.

Proof. (1) Let $V \in \mathscr{N}_0$ convex. We have to show that V is absorbing. For this, we have to show that $\forall x \in E$, there exist $\rho > 0$ such that $\lambda x \in V$, for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$.
Consider the mapping $\phi : \mathbb{K} \longrightarrow E$ defined by $\phi(\lambda) = \lambda x, \forall \lambda \in \mathbb{K}$ ($x \in E$ is fixed).

Claim: ϕ is α-continuous.

Let U be any open set in E containing $\phi(\lambda)$. Then there exist $A \in N_\lambda(\mathbb{K})$ and $B \in N_x(E)$ such that $A.B \subseteq U$. In particular, $Ax \subseteq U$; that is, $\phi(A) \subseteq U$. This reflects that ϕ is α-continuous at λ and hence ϕ is α-continuous.

Now since ϕ is α-continuous and $0 = 0.x$, there exist $U' \in N_0(\mathbb{K})$ such that $U'x \subseteq V$. Without loss of generality, we may assume that U' is convex because V is convex. Thus, there exist $\rho > 0$ such that $D = \{\lambda \in \mathbb{K} : |\lambda| \leq \rho\} \subseteq U'$ and hence $\lambda x \in V$, for all $\lambda \in D$.

(2) Let $V \in \mathscr{N}_0$ be convex. Since $0 = 0.0$, there exist $A \in N_0(\mathbb{K})$ and $B \in N_0(E)$ such that $A.B \subseteq V$. By convexity of V, we may assume that $A \in N_0(\mathbb{K})$ is convex. Consequently, there exist $\varepsilon > 0$ such that $D = \{\lambda \in \mathbb{K} : |\lambda| \leq \varepsilon\} \subseteq A$. Let $U = \bigcup_{|\lambda| \leq \varepsilon} \lambda B$. Clearly, $U \in N_0(E)$.

In order to finish the proof, we show that U is balanced. For, let $\eta \in \mathbb{K}$ with $|\eta| \leq 1$. Then $\eta U = \bigcup_{|\lambda| \leq \varepsilon} \eta \lambda B$
$= \bigcup_{|\lambda| \leq \varepsilon} |\eta| \lambda B = \bigcup_{|\gamma| \leq \varepsilon} \gamma B \subseteq U$. Thus, U is balanced.

□

Question 1. Does Theorem 2.15 hold for any $V \in \mathscr{N}_0$, i.e, Is every $V \in \mathscr{N}_0$ convex?

Theorem 2.16. *Let E be an α-topological vector space and V be any convex open set in E containing zero. Moreover, if $\{r_n\}$ is a sequence of positive real numbers such that $\lim_{n \longrightarrow \infty} r_n = \infty$, then $E = \bigcup_{n=1}^{\infty} r_n V$.*

Proof. Since V is convex, by theorem 2.15, V is absorbing. Consequently, for any $x \in E$, there exist $\varepsilon > 0$ such that $\lambda x \in V, \forall \lambda \in \mathbb{K}$ with $|\lambda| \leq \varepsilon$. Also, since $r_n \longrightarrow \infty$, for sufficiently large n, $|\frac{1}{r_n}| \leq \varepsilon \Rightarrow \frac{1}{r_n} x \in V$, for sufficiently large n and hence $E = \bigcup_{n=1}^{\infty} r_n V$. □

Corollary 2.17. *Let E be an α-topological vector space and U be a convex open set in E containing zero. Then $E = \bigcup_{n=1}^{\infty} nU$.*

Question 2. Can convexity of V in Theorem 2.16 be dropped?

Definition 2.18. *Let E be an α-topological vector space. Then a subset $A \subseteq E$ is said to be*
(1) bounded if for every $U \in \mathscr{N}_0$, there exist $r > 0$ such that $A \subseteq sU$ for all $s > r$.
(2) α-bounded if for every $U \in N_0(E)$, there exist $r > 0$ such that $A \subseteq sU$ for all $s > r$.
(3) c-bounded if for every convex $U \in \mathscr{N}_0$, there exist $r > 0$ such that $A \subseteq sU$ for all $s > r$.

Theorem 2.19. *Let E be an α-topological vector space such that $nU \in N_0(E)$, for all $U \in N_0(E), n \in \mathbb{N}$. Then every α-compact set in E is c-bounded.*

Proof. Let A be an α-compact set in E and choose any convex open set V in E containing 0. By theorem 2.15, we find a balanced $U \in N_0(E)$ such that $U \subseteq V$. Corollary 2.17 yields $A \subseteq \bigcup_{n=1}^{\infty} nU$. Since $nU \in \tau^{\alpha}$ and A is α-compact, we have $A \subseteq \bigcup_{n=1}^{N} n_k U$.
Since U is balanced, $A \subseteq n_N U$. Consequently, for any $s > n_N$, we have $A \subseteq n_N U = s.\frac{n_N}{s} U \subseteq sU \subseteq sV$. This proves that A is c-bounded. □

Theorem 2.20. *Let A and B be α-bounded subsets of an α-topological vector space E. Then $A+B$ and $A \cup B$ are c-bounded.*

Proof. Choose a convex set $V \in \mathscr{N}_0$. Then there exist a balanced $U \in N_0(X)$ such that $U+U \subseteq V$. By assumption, there exist $r_1, r_2 > 0$ such that $A \subseteq sU$ and $B \subseteq s'U$, for each $s > r_1$, $s' > r_2$.
Let $r = max\{r_1, r_2\}$. Then, for any $t > r$, we have $tV \supseteq t(U+U) = tU + tU \supseteq A+B$ reflecting that $A+B$ is c-bounded.
Similarly, since $\forall t > r, tV \supseteq tU = t(U \cup U) = (tU) \cup (tU) \supseteq A \cup B$, it follows that $A \cup B$ is c-bounded. Thus, the proof is finished. □

Theorem 2.21. *Let A and B be subsets of an α-topological vector space X such that A is α-compact, B is α-closed. Then $A+B$ is α-closed.*

Proof. Let $x \notin A+B$. Then, for all $a \in A$, $x \notin a+B$. Since $a+B$ is α-closed by theorem 2.5, there exist α-open set U_α and V_α in X such that $x \in U_\alpha, a+B \subseteq V_\alpha$ and $U_\alpha \cap V_\alpha = \emptyset$. As a consequence, $A \subseteq \bigcup_{a \in A} (V_\alpha - B)$. Notice that $(V_\alpha - B)$ is α-open in X and therefore, there exist a finite subset S of A such that $A \subseteq \bigcup_{a \in S} (V_\alpha - B)$.

Let $U = \bigcap_{a \in S} U_\alpha$. Then U is α-open in X containing x such that $U \cap (A+B) = \emptyset$. For if there is $y \in U \cap (A+B)$, then $y \in V_\alpha$ for some $a \in V_\alpha$, the absurd. Thus $x \notin \alpha Cl(A+B)$. Hence the assertion follows. □

Theorem 2.22. *Let A and B be any subsets of an α-topological vector space E. Then $\alpha Cl(A) + \alpha Cl(B) \subseteq Cl(A+B)$.*

Proof. Omitted, trivial to proof. □

Theorem 2.23. *For any subset C and D of an α-topological vector space E. Then $C + Int(D) \subseteq \alpha Int(C+D)$.*

Proof. Omitted, trivial to proof. □

Corollary 2.24. *Let C and D be any subsets of α-topological vector space E. Then $Int(C) + Int(D) \subseteq \alpha Int(C+D)$.*

3. Characterizations

In this section, we present some important and useful depths of α-topological vector spaces.

Theorem 3.1. *For a subset A of an α-topological vector space E and for each $x \in E$, the following are satisfied:*
(a) $\alpha Cl(x+A) \subseteq x + Cl(A)$.
(b) $x + \alpha Cl(A) \subseteq Cl(x+A)$.
(c) $x + Int(A) \subseteq \alpha Int(x+A)$.
(d) $Int(x+A) \subseteq x + \alpha Int(A)$.

Proof. (a) Let $y \in \alpha Cl(x+A)$ and consider $z = -x+y$ in E. Let W be any open neighborhood of z. Then we get α-open set U containing $-x$ and V containing y in E such that $U+V \subseteq W$. By assumption, we have $(x+A) \cap V \neq \emptyset$, so there is $a \in E$ such that $a \in (x+A) \cap V$. Now, $-x+a \in A \cap (U+V) \subseteq A \cap W \Rightarrow A \cap W \neq \emptyset$ and hence $z \in Cl(A)$; that is, $y \in x + Cl(A)$. Therefore, $\alpha Cl(x+A) \subseteq x + Cl(A)$.

(b) Let $z \in x + \alpha Cl(A)$. Then $z = x+y$ for some $y \in \alpha Cl(A)$. Notice that for any open neighborhood W of z, there exist α-open set $U, V \in \alpha O(E)$ such that $x \in U$ and $y \in V$ and $U+V \subseteq W$. Since $y \in \alpha Cl(A)$, $A \cap V \neq \emptyset \Rightarrow$ there is $a \in A \cap V$. Now $x+a \in (x+A) \cap (U+V) \subseteq (x+A) \cap W \Rightarrow (x+A) \cap W \neq \emptyset \Rightarrow z \in Cl(x+A)$. Hence the assertion follows.

(c) Let $y \in x + Int(A)$. Then $U+V \in Int(A)$ for some $U, V \in \alpha O(E)$ such that $-x \in U$ and $y \in V$. Whence we have $-x+V \subseteq U+V \subseteq A \Rightarrow V \subseteq x+A$. Since V is α-open, $y \in \alpha Int(x+A)$ and consequently, $x + Int(A) \subseteq \alpha Int(x+A)$.

(d) Let $y \in Int(x+A)$. Then $y = x+a$ for some $a \in A$. Since E is α-TVS, there exist $U, V \in \alpha O(E)$ such that $x \in U, a \in V$ and $U+V \subseteq Int(x+A)$. Now $x+V \subseteq U+V \subseteq Int(x+A) \subseteq x+A$ implies that $y \in x + \alpha Int(A)$. Therefore, the assertion follows.

Similarly arguments as above yield the following useful result.

□

Theorem 3.2. *For a subset A of an α-topological vector space E and any non-zero scalar λ, the following are valid:*
(a) $\alpha Cl(\lambda.A) \subseteq \lambda.Cl(A)$.
(b) $\lambda.\alpha Cl(A) \subseteq Cl(\lambda.A)$.
(c) $\lambda.Int(A) \subseteq \alpha Int(\lambda.A)$.
(d) $Int(\lambda.A) \subseteq \lambda.\alpha Int(A)$.

Theorem 3.3. *For any α-topological vector space E and for each $x, y \in E$, the following are true:*
(a) the translation mapping $T_x : E \longrightarrow E$ defined by $T_x(y) = x+y$ is α-continuous and
(b) the mapping $T_\lambda : E \longrightarrow E$ defined by $T_\lambda(x) = \lambda x$ is α-continuous, where λ is a fixed scalar.

Proof. (a) Let $y \in E$ and V be an open set in E containing $T_x(y) = x+y$. By the definition of α-topological vector spaces, we obtain $U, U' \in \alpha O(E)$ such that $x \in U$ and $y \in U'$ and $U + U' \subseteq V$ and consequently $T_x(U') \subseteq V$. This shows that T_x is α-continuous at y. Since $y \in E$ was arbitrary, it follows that T_x is α-continuous.

(b) Let $x \in E$ be an arbitrary. Let W be any open set in E containing λx. Then there exist α-open sets U in $\mathbb{K}$ containing λ and V in E containing x such that $U.V \subseteq W$. In particular, $\lambda V \subseteq W \Rightarrow T_\lambda(V) \subseteq W$ and hence T_λ is α-continuous. □

Theorem 3.4. *For an α-topological vector space E, the mapping $\Phi : E \times E \longrightarrow E$ defined by $\Phi(x,y) = x + y, \forall (x,y) \in E \times E$ is α-continuous.*

Proof. Let $(x,y) \in E \times E$ and let W be an open set in E such that $\Phi(x,y) = x + y \in W$. Then, by the definition of an α-topological vector space, there exist α-open neighborhood U and V of x and y resp. such that $U + V \subseteq W$. Since $U \times V$ is α-open in $E \times E$ (with respect to the product topology) such that $\Phi(U \times V) = U + V \subseteq W$. Hence, Φ is α-continuous at arbitrary point $(x,y) \in E \times E$. □

Theorem 3.5. *For an α-topological vector space E, the mapping $\Psi : \mathbb{K} \times E \longrightarrow E$ defined by $\Psi(\lambda, x) = \lambda.x, \forall (\lambda, x) \in \mathbb{K} \times E$ is α-continuous.*

Proof. Omitted, trivial to proof. □

Theorem 3.6. *Let E_1 be an α-topological vector space, E_2 be a topological vector space over the same field $\mathbb{K}$. Let $f : E_1 \longrightarrow E_2$ be a linear map such that f is continuous at 0. Then f is α-continuous everywhere.*

Proof. Let x be any non-zero element of E_1 and V be an open set in E_2 containing f(x). Since translation of an open set in topological vector spaces is open, $V - f(x)$ is open set in E_2 containing 0. Since f is continuous at 0, there exist an open set U in E_1 containing 0 such that $f(U) \subseteq V$. Furthermore, linearity of f implies that $f(x+U) \subseteq V$. By theorem 2.3, $x+U$ is α-open and hence f is α-continuous at x. By hypothesis, f is α-continuous at 0. This reflects that f is α-continuous. □

Corollary 3.7. *Let E be an α-topological vector space over the field $\mathbb{K}$. Let $f : E \longrightarrow \mathbb{K}$ be a linear functional which is continuous at 0. Then the set $F = \{x \in E : f(x) = 0\}$ is α-closed.*

Theorem 3.8. *Let A be any α-compact set in an α-topological vector space E. Then $x + A$ is compact, for each $x \in E$.*

Proof. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of $x + A$. Then $A \subseteq \cup_{\alpha \in \Delta}(-x + U_\alpha)$. By hypothesis and theorem 2.3, $A \subseteq \cup_{\alpha \in \Delta_0}(-x + U_\alpha)$ for some finite $\Delta_0 \subseteq \Delta$. Whence we find that $x + A \subseteq \cup_{\alpha \in \Delta_0} U_\alpha$. This shows that $x + A$ is compact. Hence the proof. □

Theorem 3.9. *Let A be any α-compact set in an α-topological vector space E. Then λA is compact, for each scalar λ.*

Proof. If $\lambda = 0$ we are nothing to prove. Assume that λ is non-zero. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of λ A. Then $A \subseteq \cup_{\alpha \in \Delta_0}(\frac{1}{\lambda} U_\alpha)$. In view of theorem 2.3, $\frac{1}{\lambda} U_\alpha$ is α-open and consequently, by hypothesis, $A \subseteq \cup_{\alpha \in \Delta_0}(\frac{1}{\lambda} U_\alpha)$ for some finite $\Delta_0 \subseteq \Delta$. Whence we find that $\lambda A \subseteq \cup_{\alpha \in \Delta_0} U_\alpha$. This proves that λA is compact. □

4. Conclusion

In this paper, we generate one of the generalization of the topological vector space called α-topological vector space. The notion is further presented along with examples and prominent properties. Additionally, significant characterization of the space is discussed intensively in the paper.

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