Uniqueness for a class of linear quadratic mean field games with common noise

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Mean field games theory is concerned with the study of differential games with:

- Exchangeable players (in a statistical sense)
- players in mean field interaction (a weak interaction)
- Infinitely many players (or a continuum of players)

PDE approach:

- Lasry-Lions (2006)
- Caines-Malhame-Huang (2006) (Nash Certainty Equivalence)
- Cardaliaguet, Gueant, ... (great contributions)

Probabilistic approach:

- Carmona-Delarue(2012)
- Bensoussan, Fischer, ... (great contributions)

Some applications

Applications:

- Mean field games and systemic risk
- Volatility formation, price formation and dynamic equilibria
- Crowd motion: mexican waves, congestion
- large population wireless power control problem
- Mean field games for marriage

N-players differential game (In $\mathbb{R}$ to fix ideas!)

- Consider the dynamics of the i^{th} player: $i \in \{1, .., N\}$

$$X_t^i = \psi^i + \int_0^t B(X_s^i, \bar{\mu}_s, \alpha_s^i)ds + \sigma W_t^i, t \in [0, T]$$

- Mean field interaction through

$$\bar{\mu}_t = \frac{1}{N}\sum_{j=1}^{N} \delta_{X_t^j}$$

- Each player wants to minimize the cost

$$J^i(\alpha^1, \alpha^2, ..., \alpha^i, ..., \alpha^N) = \mathbb{E}\Big(G(X_T^i, \bar{\mu}_T) + \int_0^T F(X_t^i, \bar{\mu}_t, \alpha_t^i)dt \Big)$$

Nash equilibrium

We say that a collection of controls $(\alpha^{1*}, ..., \alpha^{i*}, ..., \alpha^{N*})$ form a Nash equilibrum if for all $i = 1, ..., N$ we have

$$J^i(\alpha^{1*}, ..., \alpha^{i*}, ..., \alpha^{N*}) \leq J^i(\alpha^{1*}, ..., \alpha^{i}, ..., \alpha^{N*})$$

i.e Once an equilibrium is in force, no player has unilateral incentive to leave the equilibrium !!!

Why consider MFG theory?

- Finding Nash equilibria is a very complex problem for games with large number of players:
  - MFG theory allows to construct approximate Nash equilibria for such games, and error term goes to zero as $N \to \infty$.
  - MFG theory provide a decentralized way to compute approximate Nash equilibria for games with large number of players.

Approximate Nash Equilibria

- We say that a collection of controls $(\alpha^{1*}, ..., \alpha^{i*}, ..., \alpha^{N*})$ form an approximate Nash equilibrum if there exists $\epsilon_N > 0$ such that for all $i = 1, ..., N$ we have

$$J^i(\alpha^{1*}, ..., \alpha^{i*}, ..., \alpha^{N*}) \leq J^i(\alpha^{1*}, ..., \alpha^{i}, ..., \alpha^{N*}) + \epsilon_N$$

- Mean field games approach allows $\epsilon_N \to 0$ as $N \to \infty$

Mean Field Games('$N = \infty'$')

- Players are indistinguishable so that the dynamics of players can be seen as dynamics of a single representative player.
- Propagation of chaos:
  - For specified players dynamics $\bar{\mu}_t = \frac{1}{N}\sum_{j=1}^{N} \delta_{X_t^j} \to \mu_t$ (Sznitman 1991)
  - Consistency demands a similar behaviour for the players at equilibrium

MFG-solution scheme

1. (mean field input) Fix a flow of probability measures $(\mu_t)_{t\in[0,T]}$ (candidate for the mass profile at equilibrium)
2. (cost minimization) Find α^* such that

$$J(\alpha^*) = \min_{\alpha} J(\alpha) := \mathbb{E}\Big(G(X_T, \mu_T) + \int_0^T F(X_t, \mu_t, \alpha_t) dt \Big)$$

   subject to

$$X_t = \psi + \int_0^t B(X_s, \mu_s, \alpha_s) ds + \sigma W_t, t \in [0, T]$$

3. (Consistency condition) Find $(\mu_t)_{t\in[0,T]}$ such that for all $t \in [0, T]$ $\mu_t = \mathcal{L}(X_t^{\alpha^*})$

$\rightarrow (\alpha_t^*, \mu_t)_{t\in[0,T]}$ is called an MFG-solution

Probabilistic approach

- (stochastic Pontryagin principle): α^* solves cost minimization problem if there is a solution to

$$\begin{cases} dX_t = \partial_y H(X_t, Y_t, \alpha_t^*, \mu_t)dt + \sigma dW_t \\ dY_t = -\partial_x H(X_t, Y_t, \alpha_t^*, \mu_t)dt + Z_t dW_t \\ X_0 = \psi, \quad Y_T = \partial_x G(X_T, \mu_T) \end{cases}$$

  where $H(X_t, Y_t, \alpha_t^*, \mu_t) = \min_{\alpha_t} H(X_t, Y_t, \alpha_t, \mu_t)$ for all t $\mathbb{P} - a.s.$ $\rightarrow$ Forward-Backward SDEs involved
- Find μ such that for all t, $\mu_t = \mathcal{L}(X_t^{\alpha^*})$

Solvability results of MFG-solution scheme

- For $T > 0$ small, existence and uniqueness.
- Existence for $T > 0$ large via Schauder-type theorems.
- Uniqueness for $T > 0$ large via the Lasry-Lions monotonicity conditions:

$$\begin{cases} \int [F(x,m) - F(x,m')](m - m')(x)dx \geq 0 \\ \int [G(x,m) - G(x,m')](m - m')(x)dx \geq 0 \end{cases}$$

- Numerical methods available in PDE approach.
- Not much is known with common noise

Noise and uniqueness (Peano Example!)

- Consider the ODE

$$dx_t = b(x_t)dt, \;\; x_0 = 0$$

$\rightarrow$ mutliple solutions when $b(x) = sign(x)$

- Consider the SDE

$$dx_t = b(x_t)dt + \epsilon dB_t \;\; x_0 = 0$$

$\rightarrow$ unique strong solution when $b(x) = sign(x)$

- Can additional noise yield uniqueness to MFGs for $T > 0$ large ?

Linear Quadratic N-players game with common noise

- Controlled dynamics of the i^{th} player: $i \in 1, .., N$

$$X_t^i = \psi^i + \int_0^t (-X_s^i + b(\bar{u}_s) + \alpha_s^i)ds + \sigma W_t^i + \sigma_0 B_t, \quad t \in [0, T]$$

- Mean field interaction through

$$\bar{u}_t = \frac{1}{N} \sum_{j=1}^{N} X_t^j$$

- Each player wants to minimize the cost

$$J^i(\alpha^1, \alpha^2, ..., \alpha^i, ..., \alpha^N) = \mathbb{E}\Big(\int_0^T \frac{1}{2}((X_t^i + f(\bar{u}_t))^2 + (\alpha_t^i)^2)dt + \frac{1}{2}(X_T + \bar{u}_T)^2 \Big)$$

LQ-MFG-solution scheme with common noise

- (Mean field input) Consider a process $u = (u_t)_{t\in[0,T]}$ adapted to the filtration generated by B only.
- (Cost minimization) Find α^* such that

$$J(\alpha^*) = \min_{\alpha} \mathbb{E}\left(\frac{1}{2}(X_T + g(u_T))^2 + \int_0^T \frac{1}{2}((X_t + f(u_t))^2 + \alpha_t^2))dt\right)$$

under the dynamics :

$$X_t = \psi + \int_0^t (-X_s + b(u_s) + \alpha_s)ds + \sigma W_t + \sigma_0 B_t, \quad t \in [0, T]$$

- (Consistency condition) Find u such that for all $t \in [0, T]$

$$u_t = \mathbb{E}(X_t^{\alpha *} | \mathcal{F}_T^B)$$

$\rightarrow$ we remark that for all $t \in [0, T]$ $\mathbb{E}(X_t^{\alpha^*} | \mathcal{F}_t^B) = \mathbb{E}(X_t^{\alpha^*} | \mathcal{F}_T^B)$

Solving LQ-MFG-solution scheme 1

- Let $t \mapsto \eta_t$ be the unique solution to the Riccati ODE

$$\begin{cases} \dot{\eta}_t = \eta_t^2 + 2\eta_t - 1, \\ \eta_T = 1 \end{cases}$$

Proposition 1:

There exists a solution (α^*, u) to LQ-MFG-solution scheme with common noise if and only if there exists a solution to the FBSDEs

$$\begin{cases} \forall t \in [0, T] \\ du_t = (-(1+\eta_t)u_t - h_t + b(u_t))dt + \sigma_0 dB_t \\ dh_t = ((1+\eta_t)h_t - f(u_t) - \eta_t b(u_t))dt + Z_t^1 dB_t \\ \text{and} h_T = g(u_T), u_0 = \mathbb{E}[\psi] \end{cases} \tag{1}$$

Moreover,

$$\alpha_t^* = -\eta_t X_t - h_t.$$

Stochastic Pontryagin principle 1

- The Hamiltonian is given by

$$H(t,a,x,y,u) = y(-x + a + b(u)) + \frac{1}{2}a^2 + \frac{1}{2}(x + f(u))^2$$

- The cost minimization problem has a solution α^* if we can solve the FBSDEs

$$\begin{cases} dX_t = \partial_y H(t,\alpha_t^*, X_t, Y_t, u_t)dt + \sigma dW_t + \sigma_0 dB_t \\ dY_t = -\partial_x H(t,\alpha_t^*, X_t, Y_t, u_t)dt + Z_t dW_t + Z_t^0 dB_t \\ X_0 = \psi, Y_T = X_T + g(u_T), t \in [0,T]. \end{cases}$$

Subject to

$$H(t,\alpha_t^*, X_t, Y_t, u_t) = \min_{a\in\mathbb{R}} H(t,a,X_t,Y_t,u_t), \forall t \in [0,T], a.s$$

Stochastic Pontryagin principle 2

- Thanks to the strict convexity of $(x, a) \mapsto H(t, a, x, y, u)$, the cost minimization problem has a solution $\alpha^* = -Y$ if we can solve the FBSDEs

$$\begin{cases} dX_t = (-X_t - Y_t + b(u_t))t + \sigma dW_t + \sigma_0 dB_t \\ dY_t = (-X_t + Y_t - f(u_t))dt + Z_t dW_t + Z_t^0 dB_t \\ X_0 = \psi, Y_T = X_T + g(u_T), t \in [0, T]. \end{cases} \tag{2}$$

- To solve a Linear FBSDEs, we seek solutions satisying

$$Y_t = \eta_t X_t + h_t, t \in [0, T] \tag{3}$$

  - h_t an Ito process depending only on B

Solving the cost minimization 1

- We suppose that we are given a mean field input u and solve the cost minimization
- There exist a solution to (2) satisfying (3) if and only if there exist a solution

$$\begin{cases} dh_t = ((1+\eta_t)h_t - f(u_t) - \eta_t b(u_t))dt + Z_t^1 dB_t \\ h_T = g(u_T), t \in [0, T] \end{cases} \tag{4}$$

- The proof uses Ito's formula and the ansatz.

Solving McKean-Vlasov constraint 1

- Suppose that there exist a solution to (4), so that the cost minimization is solved
- There exists u satisfying $u_t = \mathbb{E}(X_t^* | \mathcal{F}_T^B)$ if and only if there exists a solution to

$$\begin{cases} du_t = (-(1+\eta_t)u_t - h_t + b(u_t))dt + \sigma_0 dB_t \\ u_0 = \mathbb{E}[\psi], t \in [0, T] \end{cases} \quad (5)$$

- The proof consists of constructing X_t from the solution to (4) and taking conditional expectation given $\mathcal{F}_T^B$

Uniqueness of LQ-MFG-solution

Proposition 2:

Suppose that $\sigma_0 > 0$ and f, b, g bounded and Lipschitz continuous. Then there exists a unique MFG-solution to the linear quadratic mean field games studied.

- Let $w_t = \exp\left(\int_t^T (1+\eta_s)ds \right)$
- Using the transformations:

$$\begin{cases} u_t^* = w_t^{-1} u_t \\ h_t^* = w_t h_t \end{cases}$$

- (1) is equivalent to:

$$\begin{cases} du_t^* = (-w_t^{-2} h_t^* + w_t^{-1} b(w_t u_t^*))dt + w_t^{-1}\sigma_0 dB_t \\ dh_t^* = (-w_t f(w_t u_t^*) - w_t \eta_t b(w_t u_t^*))dt + Z_t^2 dB_t \\ h_T^* = g(u_T^*), u_0^* = w_0^{-1}\mathbb{E}[\psi], t \in [0, T] \end{cases} \tag{6}$$

- Thanks to the hypothesis FBSDEs (6) are nondegenerate and satisfy usual theorems of existence and uniqueness for FBSDEs $\rightarrow$ existence and uniqueness of MFG-solution.

Non-uniqueness of LQ-MFGs

- Suppose $\sigma_0 = 0$
- Counter-example to uniqueness
  - Choose $f = b = \psi = 0$
  - Let $R = \int_0^T w_s^{-2} ds > 0$
  - Consider $g : \mathbb{R} \mapsto \mathbb{R}$

$$g(x) = \begin{cases} 1 \textbf{ if } x < -R \\ -x/R \textbf{ if } -R \leq x \leq R \\ -1 \textbf{ if } x > R \end{cases} \tag{7}$$

Corresponding LQ-MFG

- (Mean field input) Consider a process $u = (u_t)_{t\in[0,T]}$ adapted to the filtration generated by B only.
- (Cost minimization) Find α^* such that

$$J(\alpha^*) = \min_{\alpha} \mathbb{E}\left(\frac{1}{2}(X_T + g(u_T))^2 + \int_0^T \frac{1}{2}(X_t^2 + \alpha_t^2))dt\right)$$

under the dynamics :

$$X_t = \int_0^t (-X_s + \alpha_s)ds + \sigma W_t, \quad t \in [0, T]$$

- (Consistency condition) Find u such that for all $t \in [0, T]$

$$u_t = \mathbb{E}(X_t^* | \mathcal{F}_T^B)$$

Many LQ-MFG solutions

- Thanks to Proposition 1, there exists a MFG-solution to the previous scheme if and only if there is a solution to:

$$\begin{cases} du_t^* = (-w_t^{-2} h_t^*)dt \\ dh_t^* = Z_t^2 dB_t \\ h_T^* = g(u_T^*), u_0^* = 0, t \in [0, T] \end{cases} \tag{8}$$

- For all $A \in \mathbb{R}$ such that $-1 \leq A \leq 1$,

$$(u_t^*, h_t^*, Z_t^2)_{t\in[0,T]} = (A \int_0^t w_s^{-2} ds, A, 0)_{t\in[0,T]}$$

  are solutions to (8).
- Thus infinitely many MFG-solutions !

Further work

- (Completed) selection through zero noise limit as $\sigma_0 \to 0$
  $\to$ If time permits, give a flavour of the result.
- (work in progress) comparing selection paradigms in situations of non unique equilibriums
- (perspective) Consider higher moments mean field interaction
  $\to$for example, variance mean field

Thank you for your attention!