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Solving nonlinear Volterra integral equations by an efficient method

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Abstract
The purpose of this article is to use M-iteration method to approximate the solution a nonlinear Volterra integral equations in Banach spaces. Our results are achieved through the concept of fixed point theory. The results in this paper are new and interesting.

Keywords: Fixed point, banach space, strong convergence, nonlinear integral equation

Introduction
Fixed point theory has received massive attention for some decades now. This is as a result of its application to certain areas in applied science and engineering such as: Optimization theory, Game theory, Approximation theory, Dynamic theory, Fractals and many other subjects.
One of the first fixed point theorems is the Banach fixed point theorem. This theorem is also known as the Banach contraction principle. Banach contraction principle is important as a source of existence and uniqueness theorem in diverse branches of sciences. This theorem gives a demonstration of the unifying power of functional analytic methods and usefulness of fixed point theory.
The Banach contraction principle uses the Picard iterative method which is defined as follows:

$$\psi_{s+1} = G\psi_s, \forall s \in \mathbb{N}, \tag{1.1}$$

for contraction mappings in a complete metric space. It is well known that this principle does not hold for nonexpansive mappings since Picard iteration method fails to converge to the fixed point of nonexpansive mappings even when the existence of fixed point is guaranteed in a complete metric space.
Some many authors have constructed several iterative methods for approximating the fixed points of nonexpansive mappings and other more general classes of mappings. An efficient iterative method is one which; converges to the fixed point of an operator, has a better rate of convergence, gives data dependent result and guarantees stability with respect to G.
Some notable iterative schemes in the existing literature includes: Mann iteration [19], Ishikawa iteration [16], Noor iteration [22], Argawal *et al.* iteration [2], Abbas and Nazir iteration [1], SP iteration [23], S* iteration [15], CR iteration [8], Normal-S iteration [25], Picard-S iteration [12], Thakur iteration [31], Thakur iteration [32], M iteration [34], M* iteration [33], Garodia and Uddin iteration [10], Two-Step Mann iteration [30] and many others.
Very recently, Ullah and Arshad [34] defined M iteration scheme as follows:

$$\begin{cases} k_s = (1-r_s)\psi_s + r_s G\psi_s, \\ \quad \eta_s = Gk_s, \forall s \geq 1, \\ \qquad \psi_{s+1} = G\eta_s, \end{cases} \tag{1.2}$$

where $\{r_s\}$ is a sequence in [0,1]. The authors showed that (1.2) converges faster than several existing iterative methods.

On the other hand, several problems which arise in mathematical physics, engineering,biology, economics and etc., lead to mathematical models described by nonlinear integral equations (see [20] and the references therein). In particular, Volterra-Fredholm integral equations arise from parabolic boundary value problems, from the mathematical modeling of the spatio-temporal development of an epidemic, and from various physical and biological models (see) [21, 36].

In this article, we will use M-iterative method (1.2) to solve the following Volterra-Fredholm integral equation which have been considered by Lungu and Rus [18]:

$$u(\psi,\eta)=g(\psi,\eta,h(u(\psi,\eta)))+\int_0^{\psi}\int_0^{\eta}k(\psi,\eta,m,n,u(m,n))dmdn, \tag{1.3}$$

For all $\psi,\eta\in\Re_+$. Let $(\Omega,|.|)$ be a Banach space. Let $\tau>0$ and

$$X_\tau=\{u\in C(\Re_+{}^2\Omega)|\exists M(u)>0:|u(\psi,\eta)|e^{-\tau(\psi+\eta)}\le M(u)\}$$

We now consider Bielecki's norm on X_τ as follows:

$$\|u\|_\tau=\sup_{\psi,\eta\in\Re_+}(|u(\psi,\eta)|e^{-\tau(\psi+\eta)})$$

Obviously, $(X_\tau,\|.\|_\tau)$ is a Banach space (see) [5].

The following result which was given by Lungu and Rus [18] will be useful in proving our main result.

Theorem 1.1. [18] Suppose the following conditions are fulfilled:

(V_1) $g\in C(\Re_+{}^2\times\Omega,\Omega), K\in C(\Re_+{}^4\times\Omega,\Omega)$

(V_2) $h:X_\tau\to X_\tau$ is such that

$\exists l_h>0:|h(u(\psi,\eta))-h(v(\psi,\eta))|\le l_h\|u-v\|.e^{\tau(\psi+\eta)}$, for all $\psi,\eta\in\Re_+$ and $u,v\in X_\tau$;

(V_3) $\exists l_g>0:|g(\psi,\eta,e_1)-g(\psi,\eta,e_2)|\le l_g|e_1-e_2|$ for all $\psi,\eta\in\Re_+$ and $e_1,e_2\in\Omega$;

(V_4) $\exists l_K(\psi,\eta,m,n_1):|K(\psi,\eta,m,n,e_1)-K(\psi,\eta,m,n,e_2)|\le l_K(\psi,\eta,m,n_1)|e_1-e_2|$,

for all $\psi,\eta,m,n\in\Re_+$ and $e_1,e_2\in\Omega$;

(V_5) $l_K\in C(\Re_+{}^4,\Re_+)$ and $\int_0^{\psi}\int_0^{\eta}l_K(\psi,\eta,m,n)e^{\tau(m+n)}dmdn\le le^{\tau(\psi+\eta)}$, for all $\psi,\eta\in\Re_+$;

(V_6) $l_gl_h+l<1$.

Then, the equation (1.3) has a unique solution $z\in X_\tau$ and the sequence of successive approximations

$$u_{s+1}(\psi,\eta)=g(\psi,\eta,h(u_s(\psi,\eta)))+\int_0^{\psi}\int_0^{\eta}K(\psi,\eta,m,n,u_s(m,n))dmdn, \tag{1.4}$$

for all $s\in\mathbb{N}$ converges uniformly to z.

We now give our main result in this section.

Theorem 1.2. Let $\{\psi_s\}$ be M-iterative method defined by (1.2) with sequences $\{r_s\}$ and $\{p_s\}$ in [0,1] such that $\sum_{s=0}^{\infty} r_s = \infty$. If all the conditions (V_1) - (V_6) in theorem 8.1 are satisfied, then the equation (1.3) has a unique solution z in X_τ and the A^* iterative sequence (1.2) converges strongly to z.

Proof. Let $\{\psi_s\}$ be an iterative sequence generated by A^* iterative method (1.2) for the operator $A: X_\tau \to X_\tau$ defined by

$$A(u(\psi,\eta)) = g(\psi,\eta,h(u(\psi,\eta))) + \int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,u(m,n))dmdn \tag{1.5}$$

We will prove that $\psi_s \to 0$ as $s \to \infty$. Using (1.2), we obtain

$$\|\psi_{s+1} - z\|_\tau = \sup_{\psi,\eta\in\Re_+} (|A(\eta_s(\psi,\eta)) - A(z(\psi,\eta))| e^{-\tau(\psi+\eta)}).$$

Now,

$$\begin{aligned}
&|A(\eta_s(\psi,\eta)) - A(z(\psi,\eta))| \\
&\le |g(\psi,\eta,h(\eta_s(\psi,\eta))) - g(\psi,\eta,h(z(\psi,\eta)))| \\
&+ \left|\int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,\eta_s(m,n))dmdn - \int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,z(m,n))dmdn\right| \\
&\le l_g |h(\eta_s(\psi,\eta)) - h(z(\psi,\eta))| \\
&+ \int_0^{\psi}\int_0^{\eta} |K(\psi,\eta,m,n,\eta_s(m,n)) - K(\psi,\eta,m,n,z(m,n))dmdn| \\
&\le l_g l_h \|\eta_s - z\|_\tau e^{\tau(\psi+\eta)} + \int_0^{\psi}\int_0^{\eta} l_K(\psi,\eta,m,n)|\eta_s(m,n)) - z(m,n)|dmdn \\
&\le l_g l_h \|\eta_s - z\|_\tau e^{\tau(\psi+\eta)} + l\|\eta_s - z\|_\tau e^{\tau(\psi+\eta)} \\
&= (l_g l_h + l)\|\eta_s - z\|_\tau e^{\tau(\psi+\eta)}
\end{aligned}$$

Hence,

$$\|\psi_{s+1} - z\|_\tau \le (l_g l_h + l)\|\eta_s - z\|_\tau \tag{1.6}$$

Again from (1.2), we have

$$\|\eta_s - z\|_\tau = \sup_{\psi,\eta\in\Re_+} (|A(g_s(\psi,\eta)) - A(z(\psi,\eta))| e^{-\tau(\psi+\eta)}).$$

Now,

$$\begin{aligned}
&|A(g_s(\psi,\eta)) - A(z(\psi,\eta))| \\
&\le |g(\psi,\eta,h(g_s(\psi,\eta))) - g(\psi,\eta,h(z(\psi,\eta)))| \\
&+ \left|\int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,g_s(m,n))dmdn - \int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,z(m,n))dmdn\right| \\
&\le l_g |h(g_s(\psi,\eta)) - h(z(\psi,\eta))| \\
&+ \int_0^{\psi}\int_0^{\eta} |K(\psi,\eta,m,n,g_s(m,n)) - K(\psi,\eta,m,n,z(m,n))dmdn|
\end{aligned}$$

$$\leq l_g l_h \left\|g_s - z\right\|_\tau e^{\tau(\psi+\eta)} + \int_0^\psi \int_0^\eta l_K(\psi,\eta,m,n)\left|g_s(m,n)) - z(m,n)\right| dmdn$$
$$\leq l_g l_h \left\|g_s - z\right\|_\tau e^{\tau(\psi+\eta)} + l\left\|g_s - z\right\|_\tau e^{\tau(\psi+\eta)}$$
$$= (l_g l_h + l)\left\|g_s - z\right\|_\tau e^{\tau(\psi+\eta)}$$

Hence,

$$\left\|\eta_s - z\right\|_\tau \leq (l_g l_h + l)\left\|g_s - z\right\|_\tau \tag{1.7}$$

Finally,

$$\begin{aligned}\left\|k_s - z\right\|_\tau &= \left\|((1-r_s)\psi_s + r_s A\psi_s) - z\right\|_\tau \\ &= \left\|((1-r_s)(\psi_s - z) + r_s(A\psi_s - z)\right\|_\tau \\ &\leq (1-r_s)\left\|\psi_s - z\right\|_\tau + r_s\left\|A\psi_s - z\right\|_\tau\end{aligned} \tag{1.8}$$

Now,

$$\left\|A\psi_s - Az\right\|_\tau = \sup_{\psi,\eta\in\Re_+} (\left|A(\psi_s(\psi,\eta)) - A(z(\psi,\eta))\right| e^{-\tau(\psi+\eta)})$$

And

$$\left|A(\psi_s(\psi,\eta)) - A(z(\psi,\eta))\right|$$
$$\leq \left|g(\psi,\eta,h(\psi_s(\psi,\eta))) - g(\psi,\eta,h(z(\psi,\eta)))\right|$$
$$+\left|\int_0^\psi \int_0^\eta K(\psi,\eta,m,n,\psi_s(m,n))dmdn - \int_0^\psi \int_0^\eta K(\psi,\eta,m,n,z(m,n))dmdn\right|$$
$$\leq l_g \left|h(\psi_s(\psi,\eta)) - h(z(\psi,\eta))\right|$$
$$+\int_0^\psi \int_0^\eta \left|K(\psi,\eta,m,n,\psi_s(m,n)) - K(\psi,\eta,m,n,z(m,n))dmdn\right|$$
$$\leq l_g l_h \left\|\psi_s - z\right\|_\tau e^{\tau(\psi+\eta)} + \int_0^\psi \int_0^\eta l_K(\psi,\eta,m,n)\left|\psi_s(m,n)) - z(m,n)\right| dmdn$$
$$\leq l_g l_h \left\|\psi_s - z\right\|_\tau e^{\tau(\psi+\eta)} + l\left\|\psi_s - z\right\|_\tau e^{\tau(\psi+\eta)}$$
$$= (l_g l_h + l)\left\|\psi_s - z\right\|_\tau e^{\tau(\psi+\eta)}$$

Thus,

$$\left\|A\psi_s - Az\right\|_\tau \leq (l_g l_h + l)\left\|\psi_s - z\right\|_\tau \tag{1.9}$$

From (1.8) and (1.9), we obtain

$$\begin{aligned}\left\|k_s - z\right\|_\tau &\leq (1-r_s)\left\|g_s - z\right\| + r_s(l_g l_h + l)\left\|g_s - z\right\|_\tau \\ &= \left[1 - r_s\left\{1 - \left(l_g l_h + l\right)\right\}\right]\left\|\psi_s - z\right\|_\tau\end{aligned} \tag{1.10}$$

By (1.6), (1.7) and (1.10), we have

$$\left\|\psi_{s+1} - z\right\|_\tau \leq (l_g l_h + l)^2\left[1 - r_s\left\{1 - (l_g l_h + l)\right\}\right]\left\|\psi_s - z\right\|_\tau$$

Recalling from assumption (C_6) that $l_g l_h + l < 1$. Thus, from (1.10), we obtain

$$\|\psi_{s+1} - z\|_\tau \leq \left[1 - r_s \left\{1 - (l_g l_h + l)\right\}\right] \|\psi_s - z\|_\tau \tag{1.11}$$

Inductively, from (1.10), we have

$$\|\psi_{s+1} - z\|_\tau \leq \|\psi_0 - z\|_\tau \prod_k^s \left[1 - r_k \left\{1 - (l_g l_h + l)\right\}\right] \tag{1.12}$$

Since $r_k \in [0,1]$ for all $k \in \mathbb{N}$ and assumption (C_6) gives

$$1 - r_k \left\{1 - (l_g l_h + l)\right\} < 1$$

From classical analysis, we know that $1 - \psi \leq e^{-\psi}$ for all $\psi \in [0,1]$. Thus, (1.12) becomes

$$\|\psi_{s+1} - z\|_\tau \leq \|\psi_0 - z\|_\tau \, e^{-\left[1 - r_k \left\{1 - (l_g l_h + l)\right\}\right] \sum_{k=0}^{s} r_k}$$

which yields $\lim_{s \to \infty} \|\psi_s - z\|_\tau = 0$. This completes the proof.

Conclusion

In this paper, the concept of fixed point has been employed to approximate the solution of a nonlinear Volterra integral equation. The method used in our results is well known to be efficient. Our results complement, improve and generalize several results in this research direction.

Availability of data and material

The data used to support the findings of this study are included within the article.

Conflicts of interest

The authors declare no conflict of interests.

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